Lecture 9: Tue Sep 15, 2020

Reminder:

- HW4 due Thursday.

Lecture

- review frequency response
- LTI systems: sinusoid in, sinusoid out

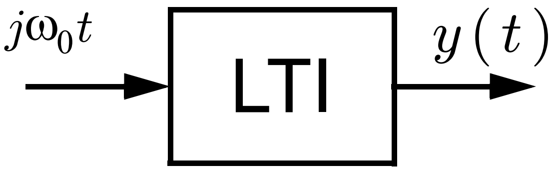
Frequency Response

(Chapter 10 of 2026 1st edition.)

10 Frequency Response 285

Complex Exponentials are Eigensignals of LTI Systems

$$x(t) = e^{j\omega_0 t}$$



$$y(t) = H(j\omega_0)\, e^{j\omega_0 t}$$

Scaling constant (eigenvalue) is:

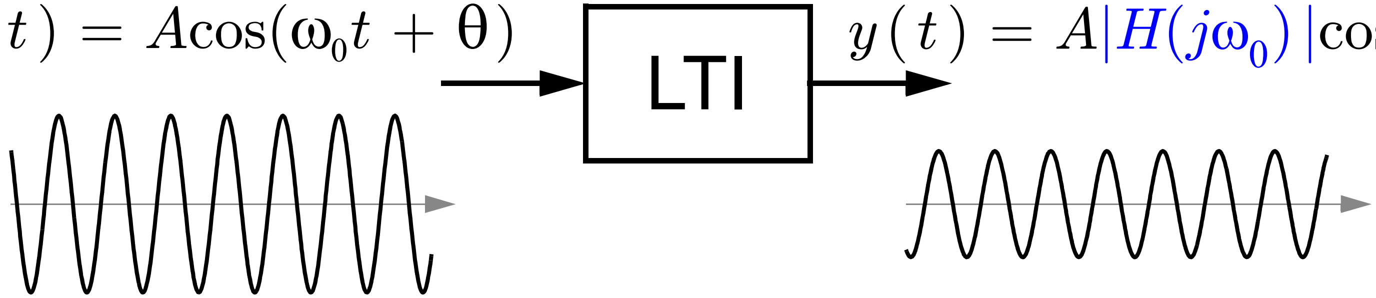
$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

"frequency response"

evaluated at ω_0.

Sinusoid-In, Sinusoid Out

$$x(t) = A\cos(\omega_0 t + \theta) \qquad y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \theta + \varphi)$$



An LTI system does only 2 things to a sinusoid with frequency ω_0:

- ▷ it attenuates amplitude by $|H(j\omega_0)|$
- ▷ it shifts phase by $\varphi = \text{angle}\{H(j\omega_0)\}$

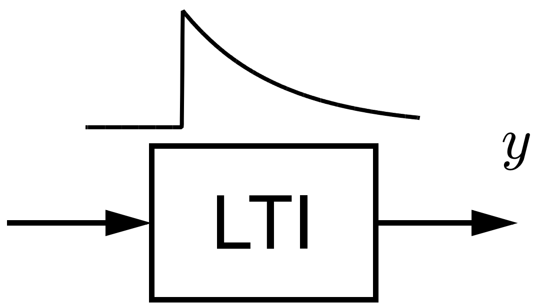
Notation:

- ▷ $H(j\omega)$ = frequency response
- ▷ $|H(j\omega)|$ = magnitude response
- ▷ $\text{angle}\{H(j\omega)\}$ = phase response

Numeric Example

Find output $y(t)$ when $x(t) = \cos(2\pi t)$ is input to
an LTI system with impulse response $h(t) = 2e^{-2t}u(t)$:

$x(t) = \cos(2\pi t)$



$y(t) = ?$

Evaluate at $\omega = 2\pi$

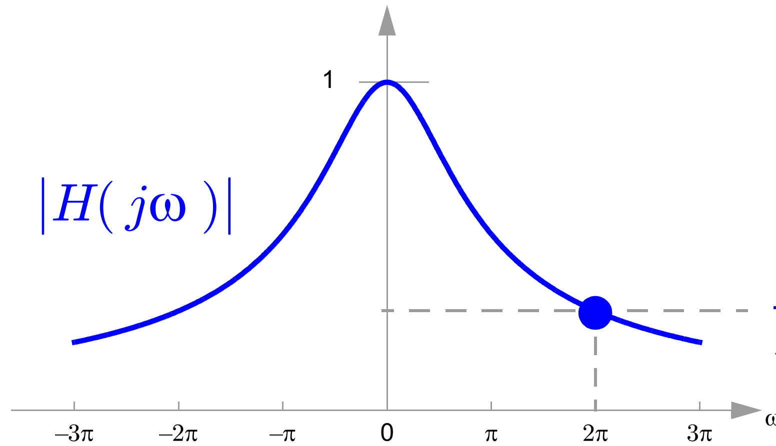
Integrate

$$\Rightarrow H(j\omega) = \frac{2}{2 + j\omega}:$$

$$\Rightarrow H(j2\pi) = \frac{1}{1 + j\pi}$$

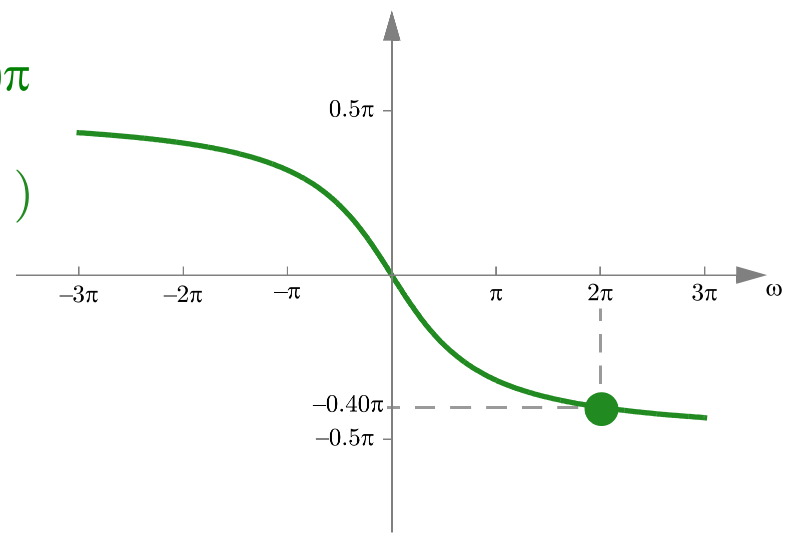
$$\approx 0.303 e^{-j0.40\pi}$$

$|H(j\omega)|$

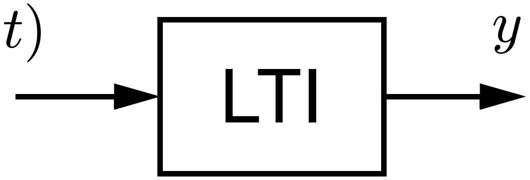


$$\frac{1}{\sqrt{1+\pi^2}} \approx 0.303$$

$\arg H(j\omega)$



$$x(t) = \cos(2\pi t)$$



$$y(t) = |H(j2\pi)| \cos(2\pi t + \varphi)$$

$$\approx 0.303 \cos(2\pi t - 0.40\pi)$$

Hermitian (Conjugate) Symmetry

Fact: If $h(t)$ is real, then $H(-j\omega) = H^*(j\omega)$

Why?

Look at definition: $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$.

▷ The LHS of equality replaces ω by $-\omega$ in integral.

▷ The RHS of equality replaces j by $-j$ in integral.

▷ Effect on the integral is the same $\Rightarrow$ LHS $=$ RHS.

Examples

(1) $$H(j\omega) = \frac{1}{\omega + j} \quad \Rightarrow \quad h(t) \text{ real?}$$

(2) $$H(j\omega) = \frac{j}{\omega + j} \quad \Rightarrow \quad h(t) \text{ real?}$$

Pop Quiz: What is the Frequency Response of a Differentiator?

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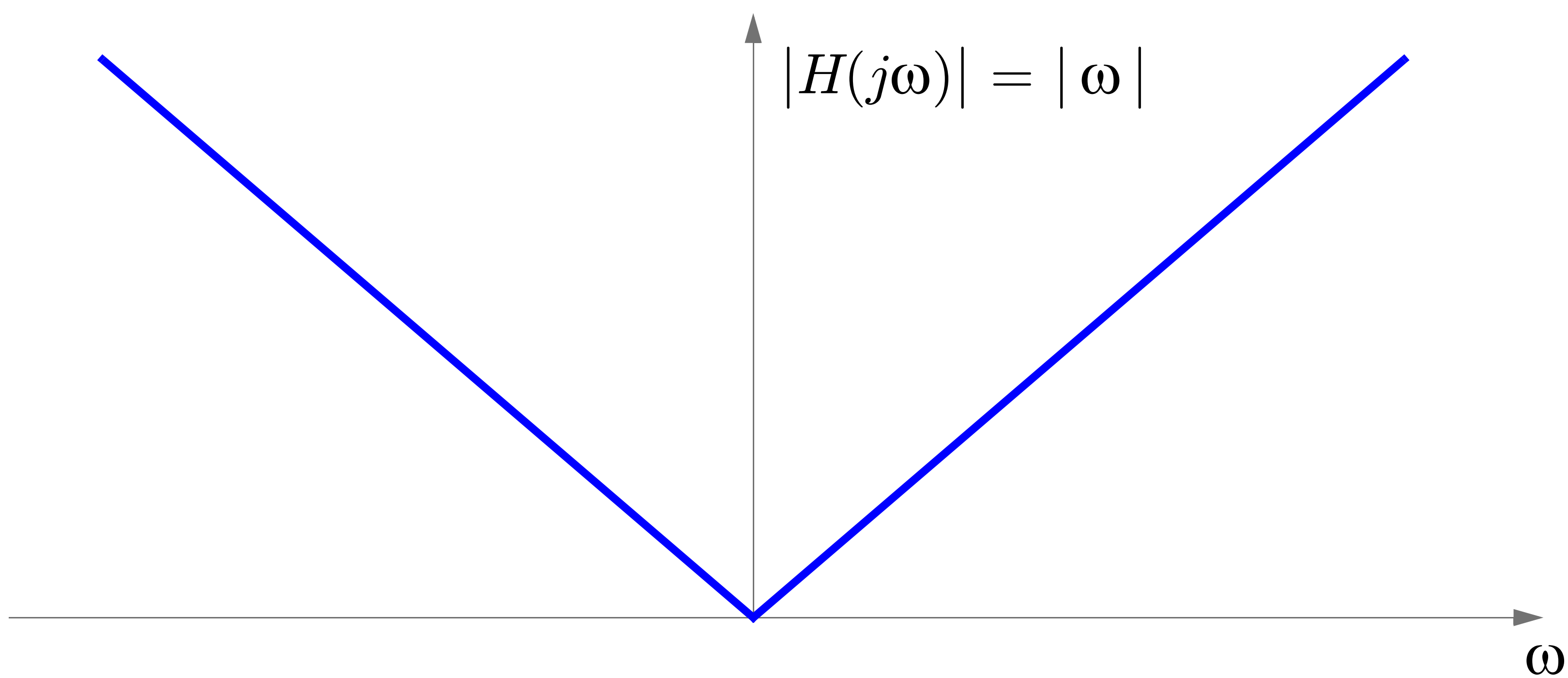
(Finding $h(t)$ directly is not easy. Exploit instead that differentiator is LTI, and that freq response is simply eigval that goes along with eigsignal $e^{j\omega t}$.)

$\Rightarrow H(j\omega) = j\omega.$

Sketch magnitude response:



Pop Quiz: What is the Frequency Response of a Delay System?

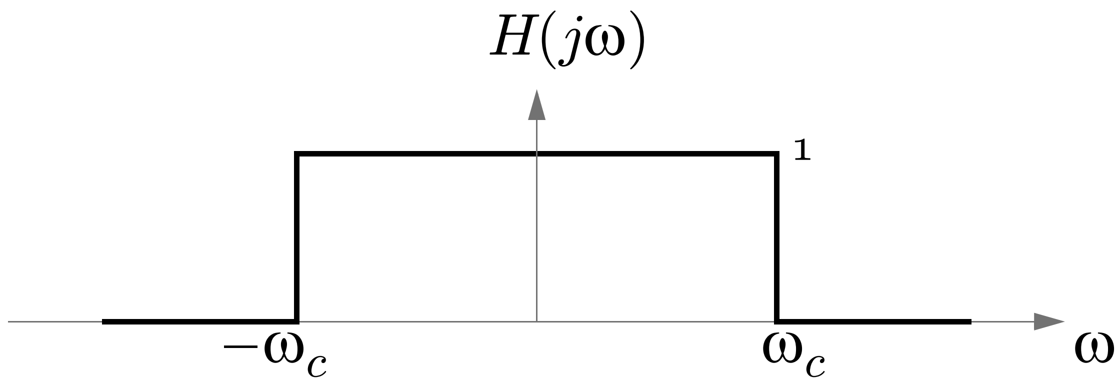
Delay by t_0

Pop Quiz: What is the Frequency Response of a Delay System?

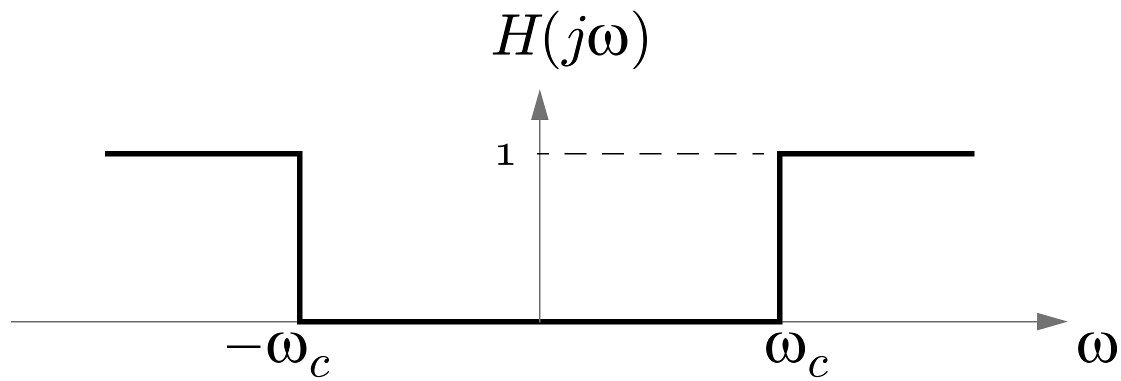
Delay by t_0

$$\Rightarrow H(j\omega) = e^{-j\omega t_0}.$$

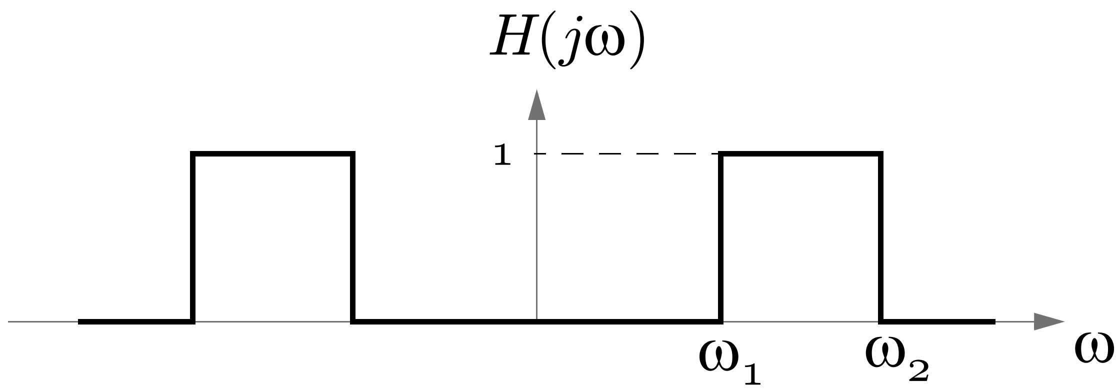
Ideal LPF = Low-Pass Filter



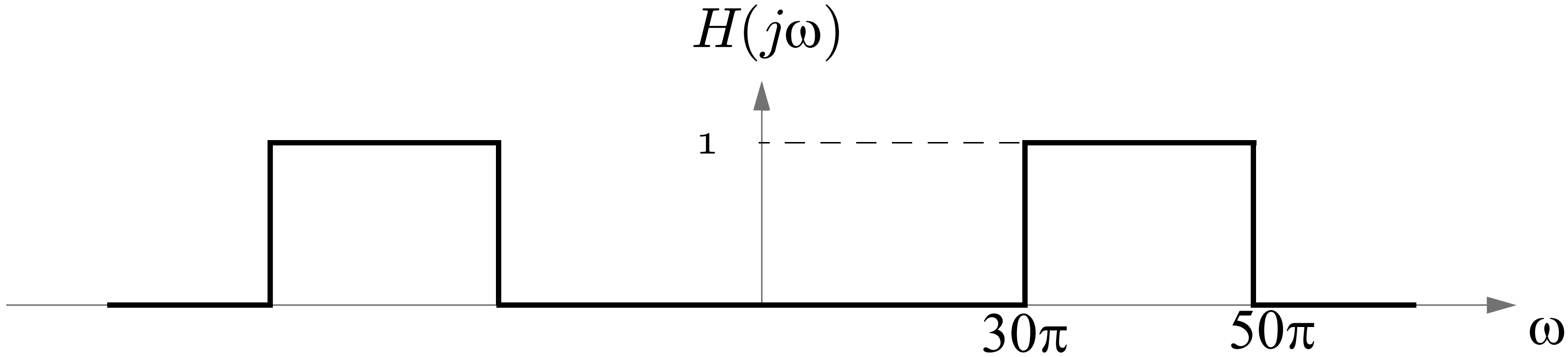
Ideal HPF = High-Pass Filter



Ideal BPF = Band-Pass Filter



Pop Quiz:

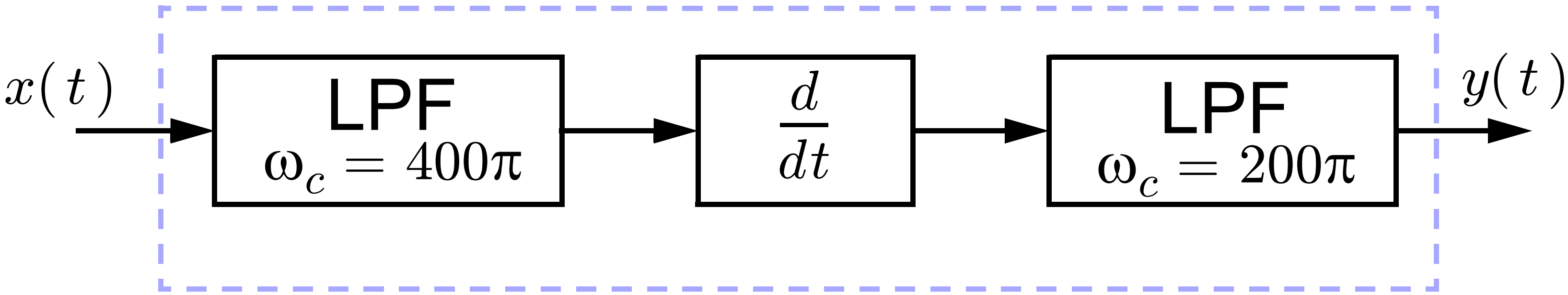


(a) Construct using a pair of *low*-pass filters.

(b) Construct using one *low*-pass and one *high*-pass filter.

(c) Construct using a pair of *high*-pass filters.

Serial Cascade of 3 Systems

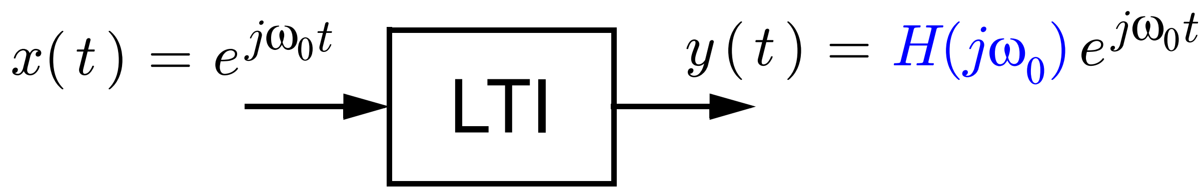


Suppose input is

$$x(t) = 17 + \cos(100\pi t) + 3\cos(300\pi t) + 5\cos(500\pi t).$$

Find overall output $y(t)$.

Importance of Sine-In, Sine Out



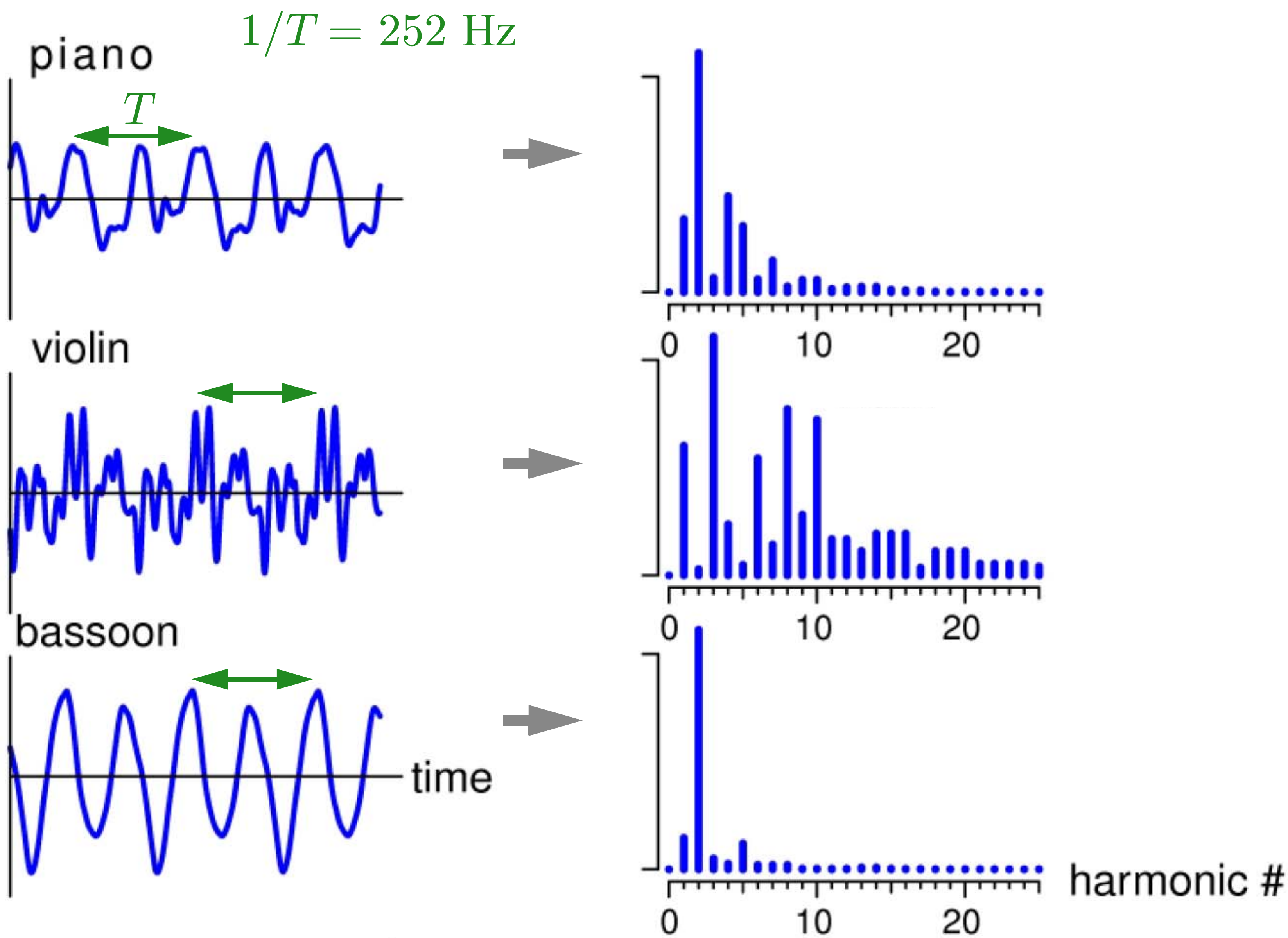
So what?

How does this help us when the input is *not a complex exponential*?

Turns out a HUUUGE class of signals can be written as sum of complex exps.

One example: Periodic signals.

Musical Instruments



Periodic Signals

A periodic signal satisfies $x(t) = x(t + T)$ for all t.

The smallest nonzero T that works is the *fundamental* period.

Key fact from 2026:

Any (!) periodic signal can be written as a sum of sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T} \qquad \text{“FS synthesis”}$$

with *harmonically* related frequencies, whose amplitudes and phases are determined by the Fourier series coefficients:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk2\pi t/T} dt \qquad \text{“FS analysis”}$$

Any interval of length T,
e.g. $[0, T)$, $[-T/2, T/2)$, etc.

Why?

Complex exponentials with different harmonic frequency are orthogonal:

$$\frac{1}{T}\int_T e^{jk2\pi t/T} e^{-jm2\pi t/T} dt = \delta[k-m] = \begin{cases} 1, \text{ when } k = m \\ 0, \text{ when } k \neq m \end{cases}$$

To derive FS analysis equation:

Multiply both sides of FS synthesis equation by $e^{-jm2\pi t/T}$,
integrate both sides over one period, and use above orthogonality result.

Fun FS Facts

- The zero-th "DC"coefficient is often computed separately: $a_0 = \frac{1}{T}\int_T x(t)dt$. Can interpret as the "average" value.

- If $x(t)$ is real then $a_{-k} = a_k^*$ $\Rightarrow$ "Hermitian" symmetry.

- If $x(t)$ is (real and) even, a_k are purely real (and even):

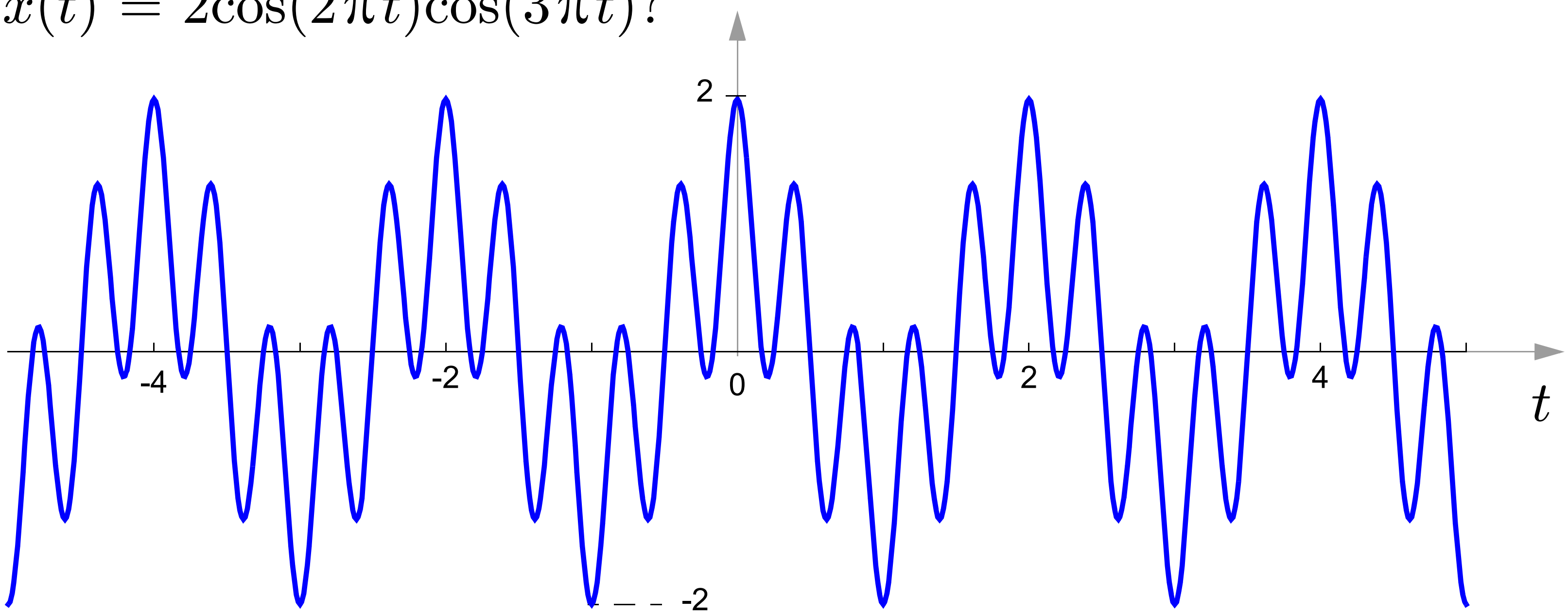
$$a_k = \frac{1}{T}\int_T x(t)\big(\cos(k2\pi t/T) - j\sin(k2\pi t/T)\big)dt$$

(even)(odd) $=$ odd, integrates to 0

- If $x(t)$ is odd, a_k are purely imaginary. E.g., consider FS for $\sin(200\pi t)$.
- Integration not always needed to find FS coefficients.

Analysis Not Always Necessary

$x(t) = 2\cos(2\pi t)\cos(3\pi t)?$



Questions:

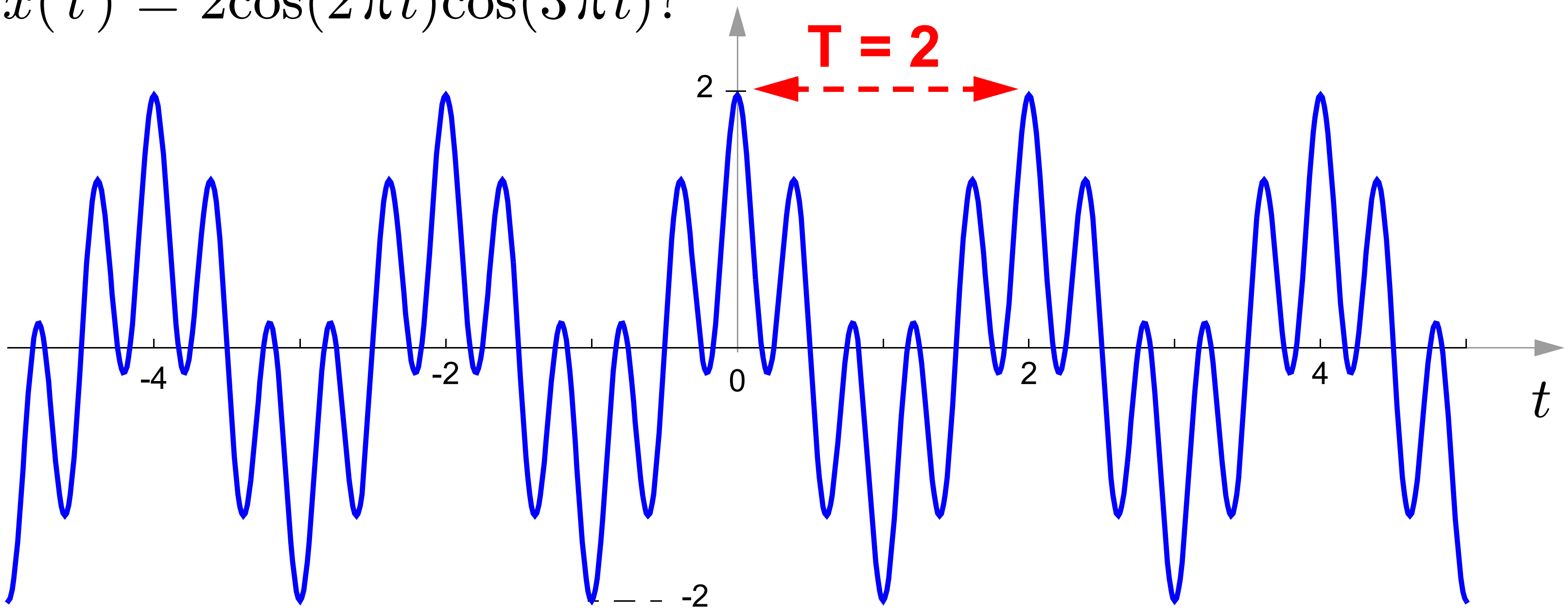
(a) Is it periodic?

(b) Find period T

(c) Find FS coefficients

Analysis Not Always Necessary

$x(t) = 2\cos(2\pi t)\cos(3\pi t)$?



Use trig ID $2\cos x \cos y = \cos(x-y) + \cos(x+y)$, and Euler:

$$\Rightarrow x(t) = \cos(\pi t) + \cos(5\pi t)$$

$$= \tfrac{1}{2}e^{j\pi t} + \tfrac{1}{2}e^{-j\pi t} + \tfrac{1}{2}e^{j5\pi t} + \tfrac{1}{2}e^{-j5\pi t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}, \qquad \text{where } T = 2,\ a_{\pm 1} = a_{\pm 5} = \tfrac{1}{2}.$$

(Derive T from this equation, use picture only to confirm that $T = 2$ makes sense.)