Lecture 7: Tue Sep 8, 2020

Reminder:

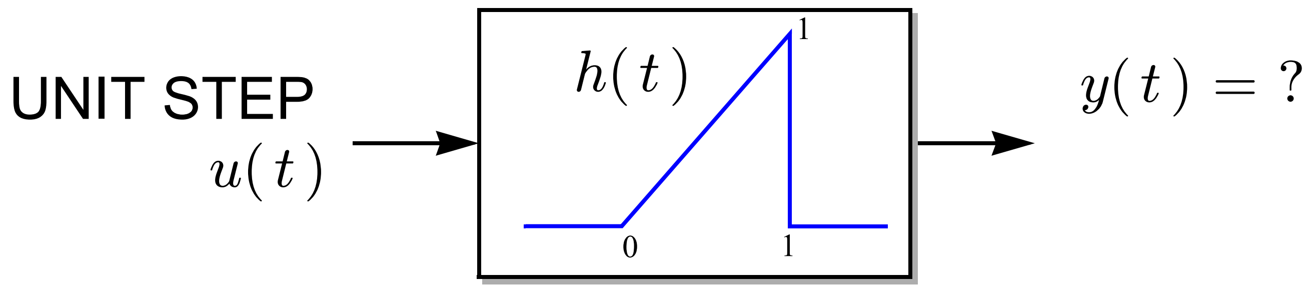
- HW3 due midnight Thursday.

Lecture

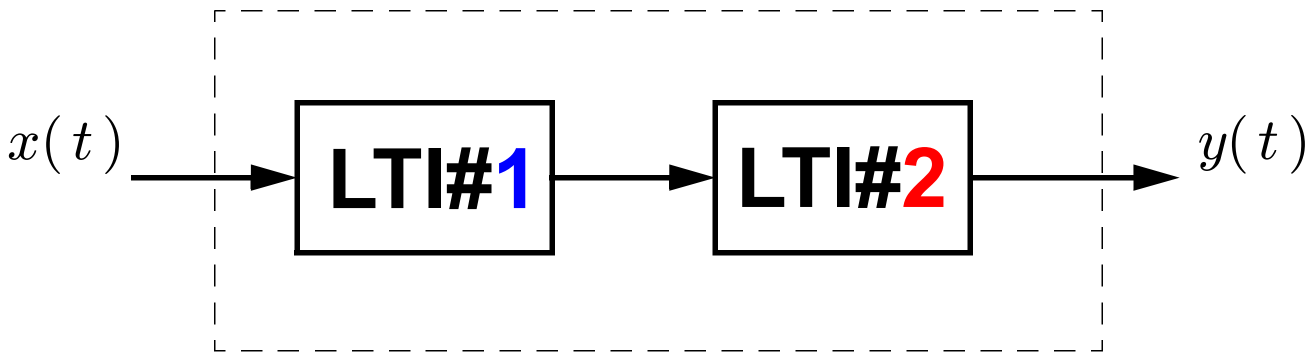
- LTI systems in cascade
- Extracting properties of LTI systems from $h(t)$
- The RADAR problem:
    - ▹ motivation
    - ▹ crosscorrelation function
    - ▹ autocorrelation function
    - ▹ matched filter

Example

Find the "step response" of a filter whose impulse response is the triangular pulse $h(t) = t(u(t) - u(t-1))$, as shown below:

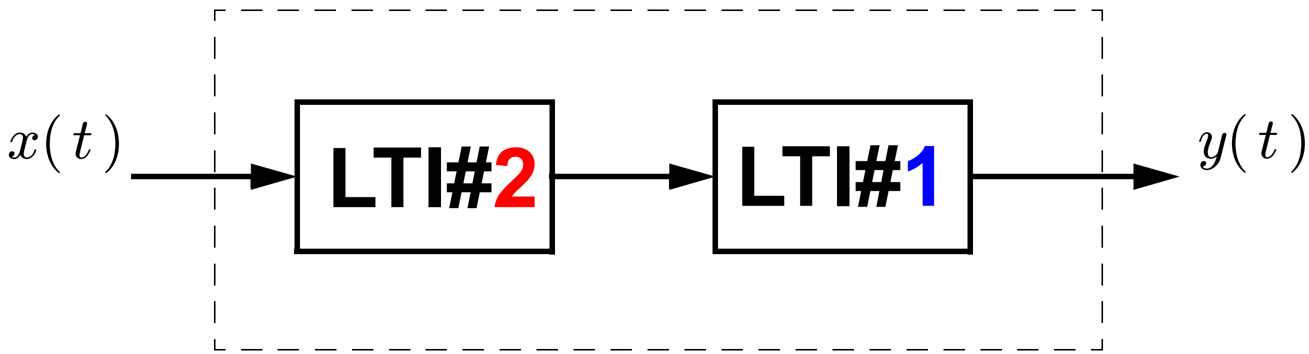


Serial Cascade

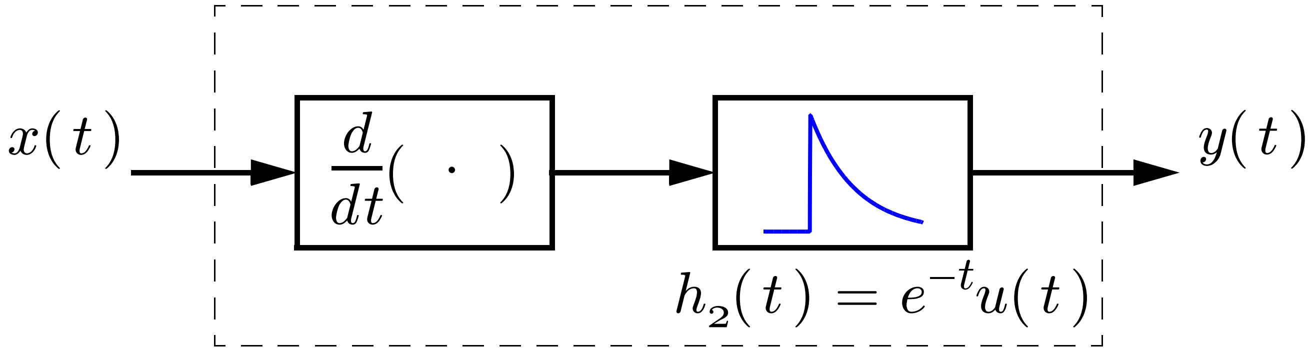


- "overall" (dashed box) system is LTI. (Why?)
- overall impulse response is $h_1(t) * h_2(t)$.

since convo is commutative, can swap! the order:

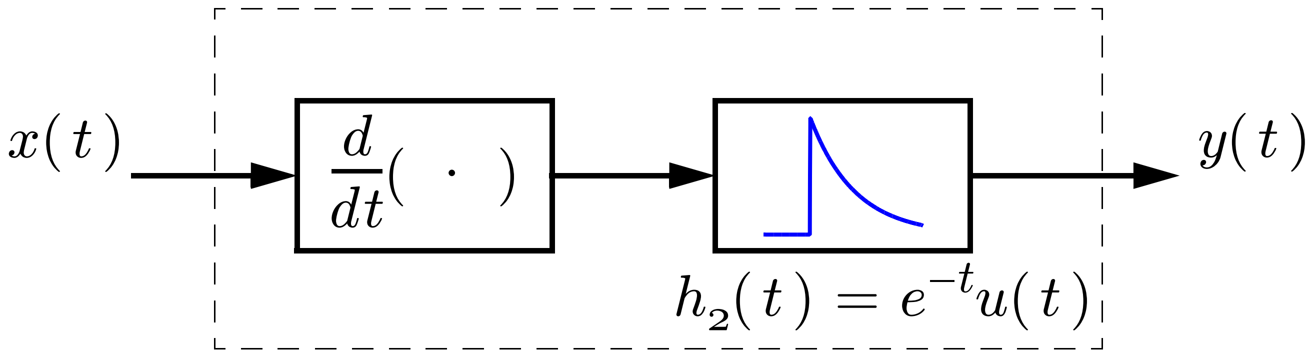


Pop Quiz #1: Find "Overall" h(t)



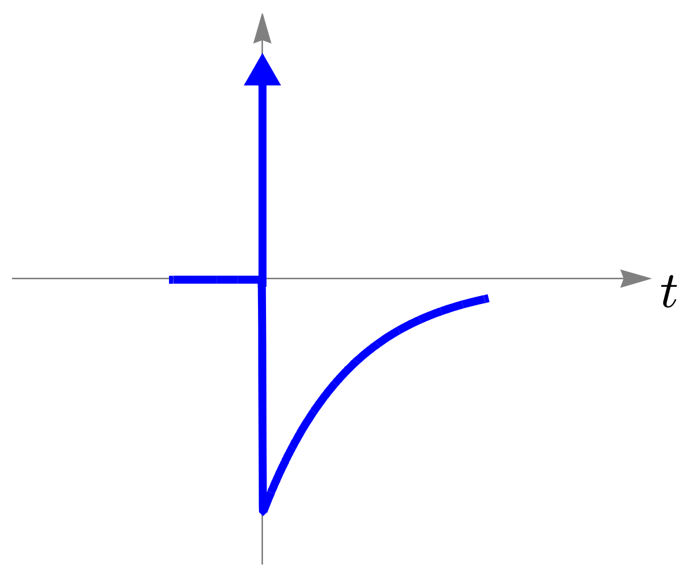
Direct approach problematic: How to differentate $\delta(t)$?

Pop Quiz #1: Find “Overall” h(t)



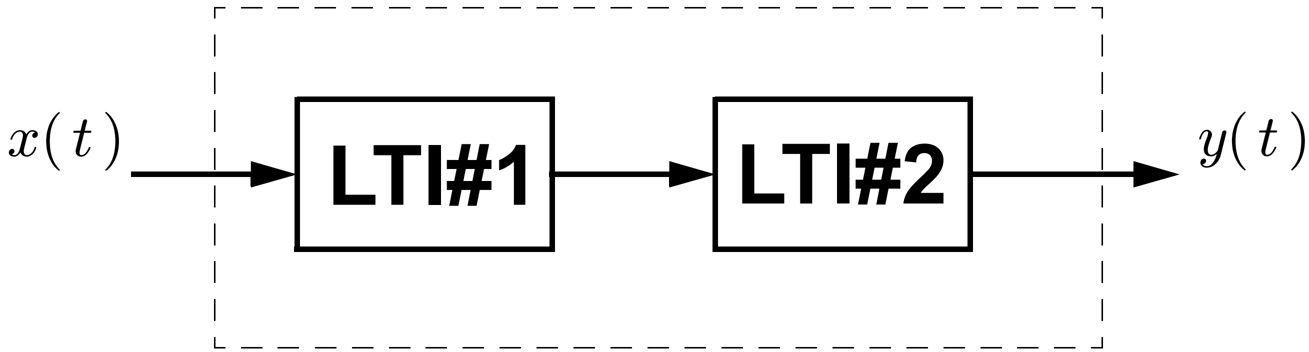
Direct approach problematic: How to differentate $\delta(t)$?

Better: Swap order! $\Rightarrow$ overall $h(t) = \frac{d}{dt} h_2(t) = \delta(t) - e^{-t}u(t)$



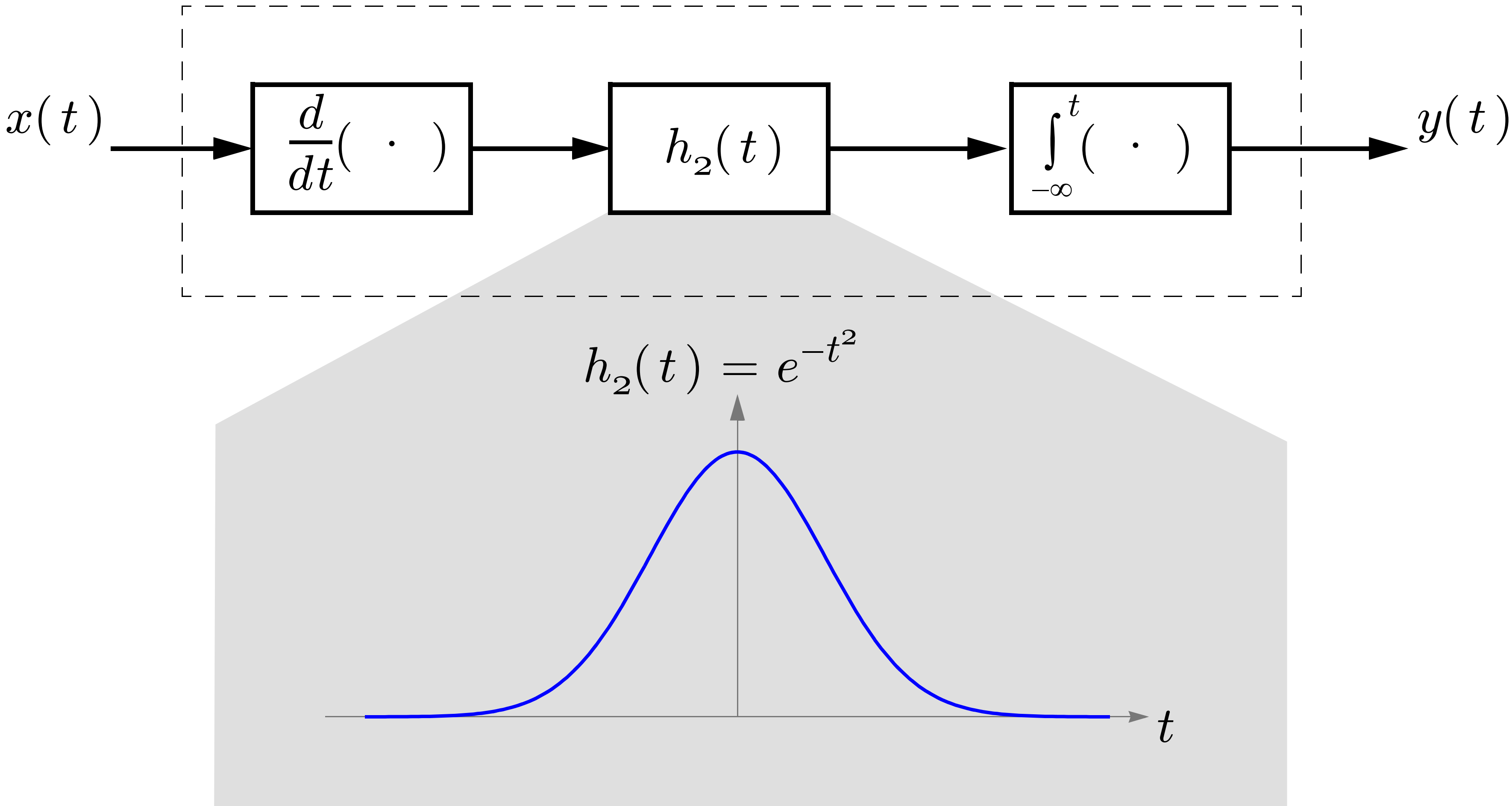
Pop Quiz #2

If LTI#1 and LTI#2 are each stable, is the serial cascade stable too?

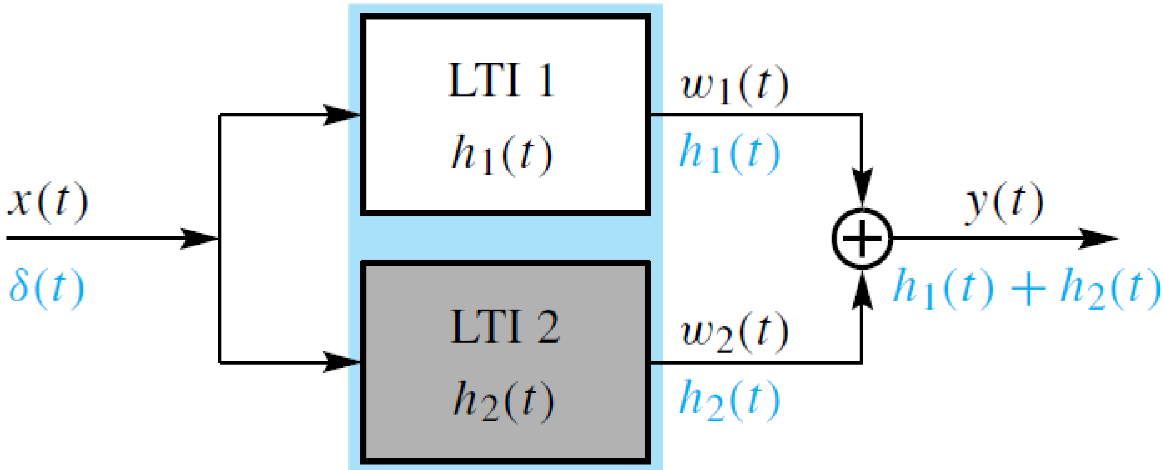


A bdd input implies a bdd intermediate signal, which in turn implies a bdd overall output.

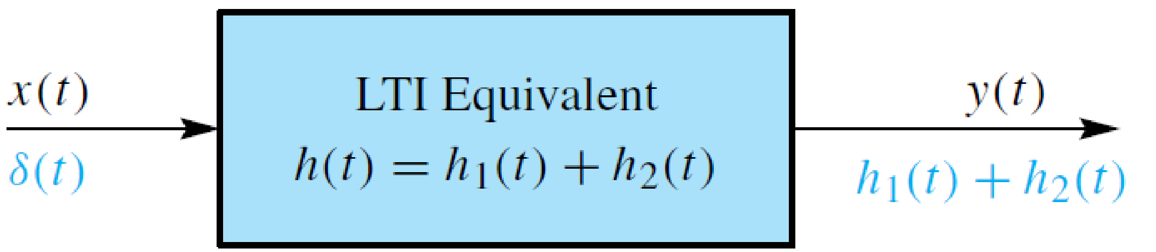
Pop Quiz #3: Find Overall h(t)



Parallel Cascade



(a)



(b)

LTI Systems: Implications on h(t)?

- memoryless $\Leftrightarrow$ $h(t) = ?$
- causal $\Leftrightarrow$ $h(t) = ?$
- stable $\Leftrightarrow$ $h(t) = ?$

LTI Systems: Implications on h(t)?

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- stable $\Leftrightarrow$ $h(t) = ?$

LTI Systems: Implications on h(t)?

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- causal $\Leftrightarrow$ $h(t) = 0 \text{ for } t < 0$
- stable $\Leftrightarrow$ $h(t) = ?$

LTI Systems: Implications on h(t)?

- memoryless $\Leftrightarrow$ $h(t) = a\delta(t)$
- causal $\Leftrightarrow$ $h(t) = 0$ for $t < 0$
- stable $\Leftrightarrow$ absolutely integrable, $\int_{-\infty}^{\infty} \left|h(t)\right| \mathrm{d}t < \infty$

BIBO stable $\Leftrightarrow$ $\int_{-\infty}^{\infty} |h(t)| \mathrm{d}t < \infty$

BIBO stable

$\Rightarrow$ the bdd input $x(t) = \mathrm{sign}(h(-t))$ must yield a bdd output at time 0:

$$y(0) = \int_{-\infty}^{\infty} x(\tau)h(-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} |h(-\tau)|\mathrm{d}\tau < \infty.$$

$\int_{-\infty}^{\infty} |h(t)| \mathrm{d}t < \infty$

$\Rightarrow$ a bounded input $|x(t)| < B$ results in an output satisfying

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau \right|$$

$$\leq \int_{-\infty}^{\infty} B \left| h(t) \right| \mathrm{d}\tau < \infty \qquad \Rightarrow \text{bounded output}$$

BIBO stable $\Leftrightarrow$ $\int_{-\infty}^{\infty} \left|h(t)\right| \mathrm{d}t < \infty$

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$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t < \infty$

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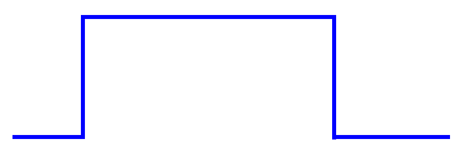
$$\leq \int_{-\infty}^{\infty} B\left|h(t)\right|\mathrm{d}\tau < \infty \qquad \Rightarrow \text{bounded output}$$

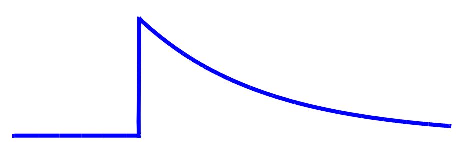
Which are stable?

▷ Is an integrator stable?

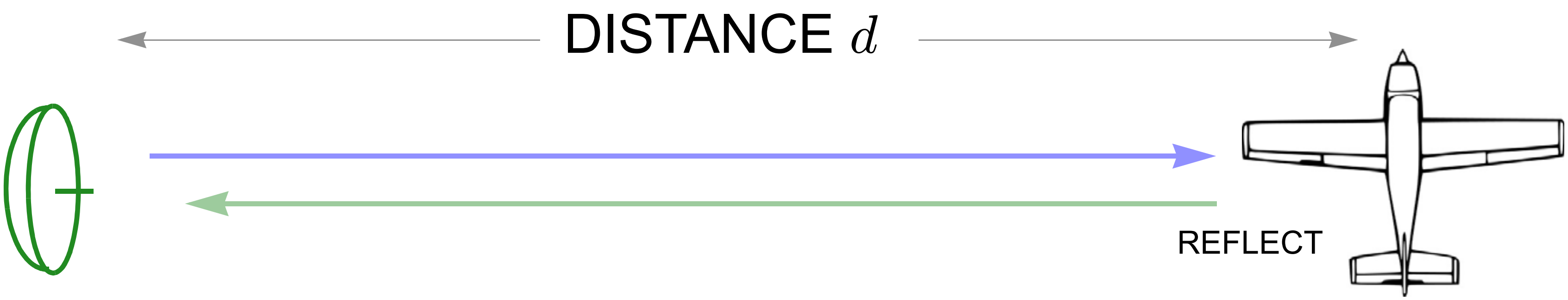
▷ Is an LTI system with rectangular impulse response stable?



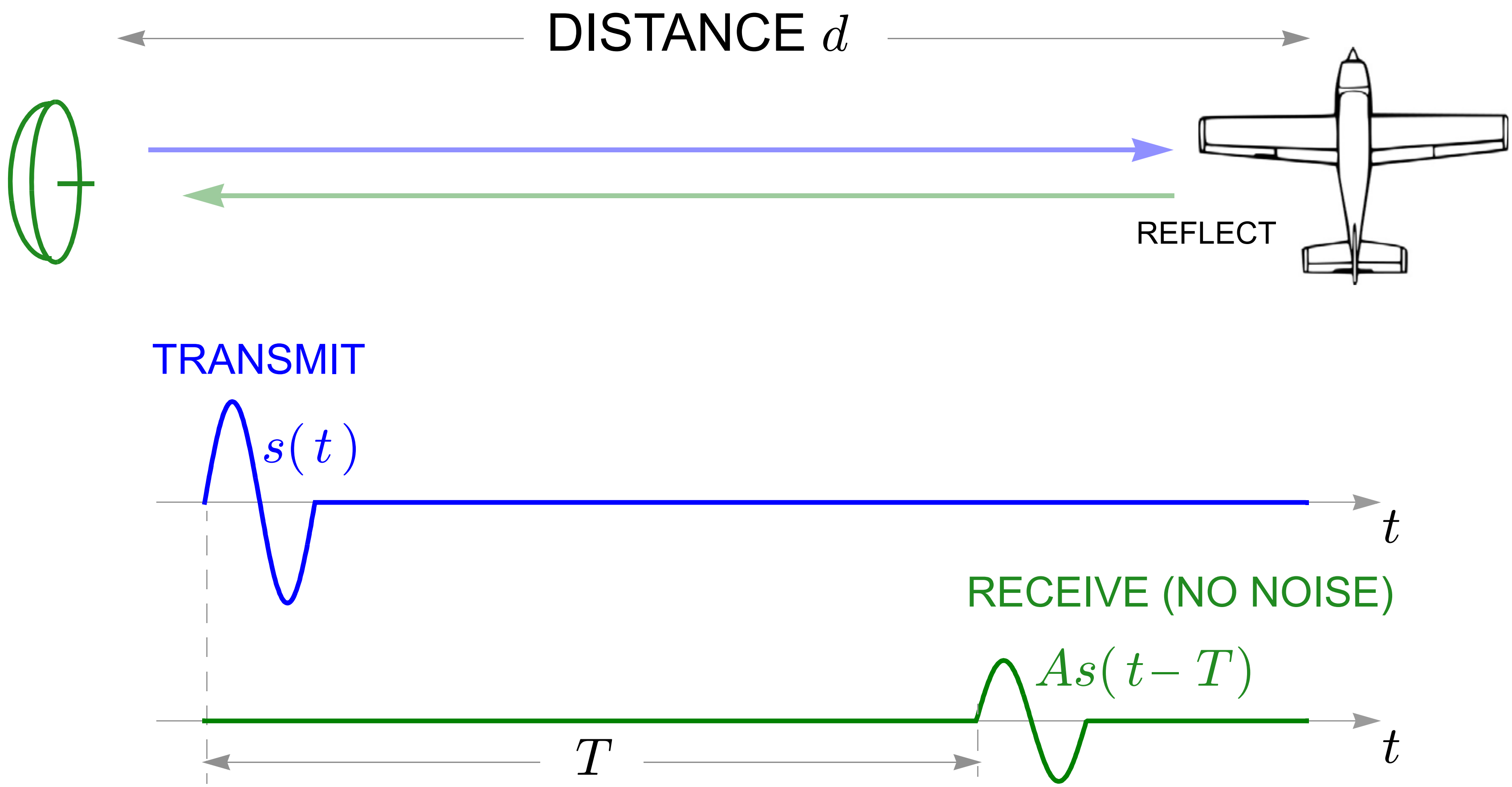
▷ Is an LTI system with $h(t) = e^{-t}u(t)$ stable?



Radar (SONAR, GPS, etc)



Radar (SONAR, GPS, etc)

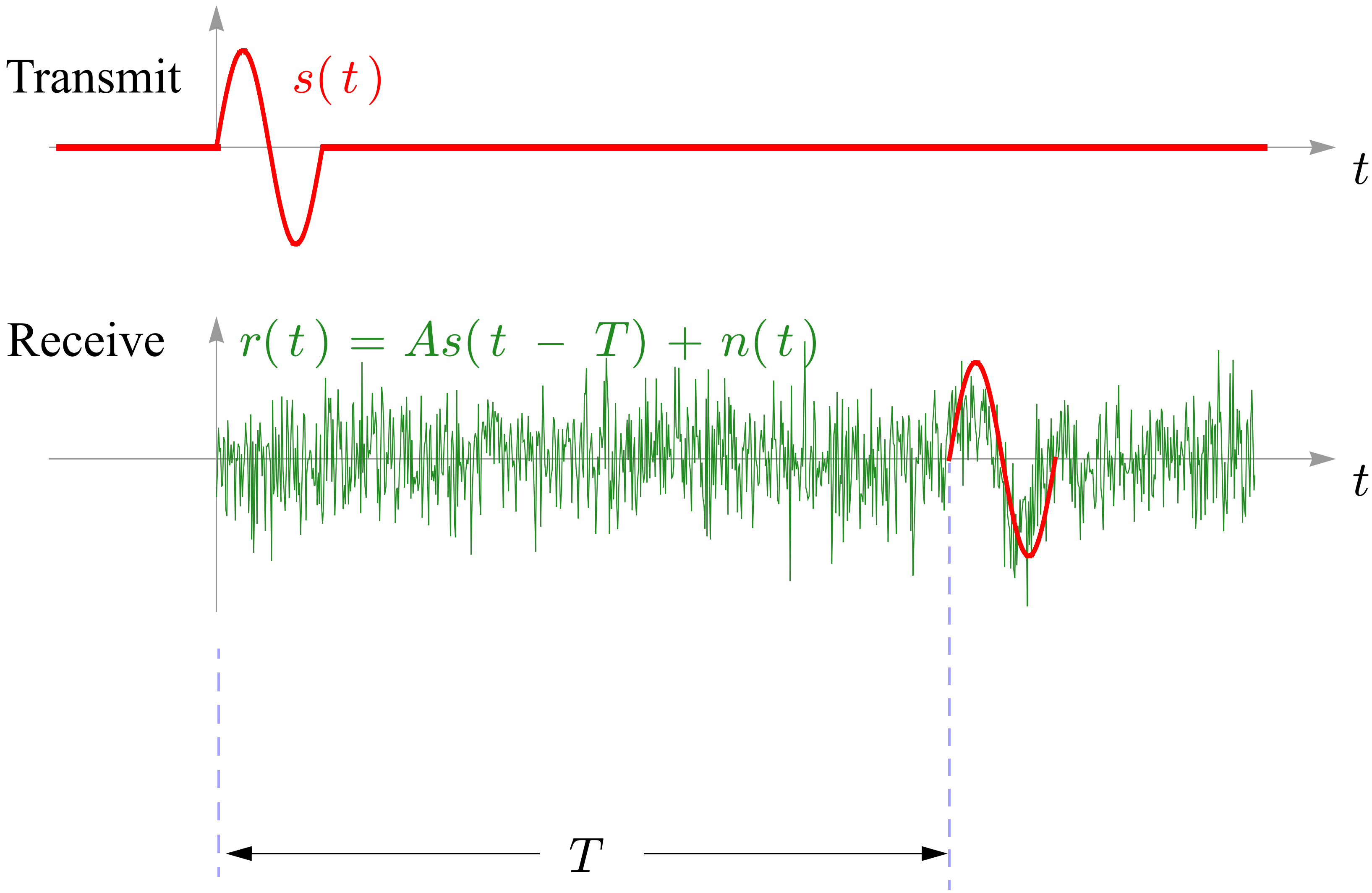


RADAR = radio detection & ranging.

Distance (range) related to delay by $d = cT/2$

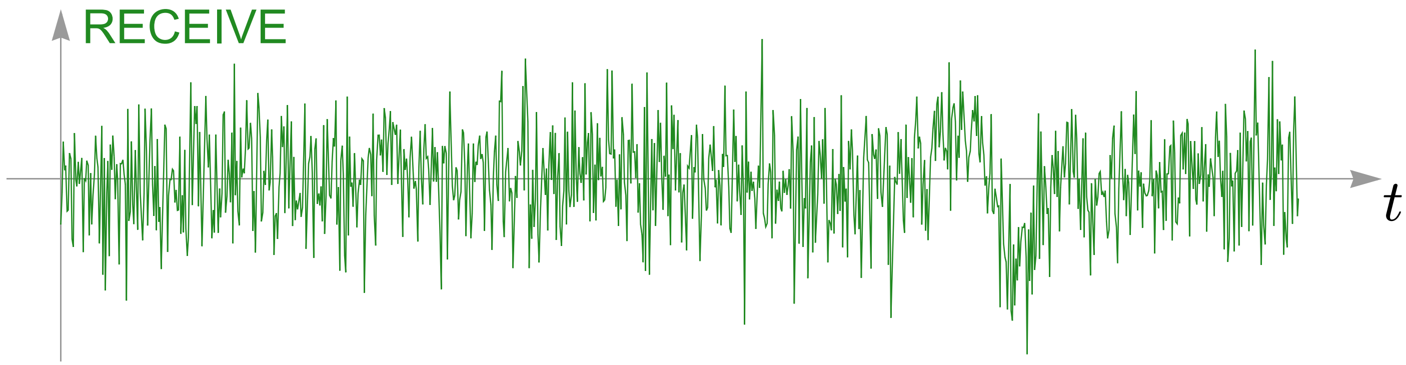
Example: How far away is airplane if roundtrip delay is $T = 1$ µs?

Impact of Noise?

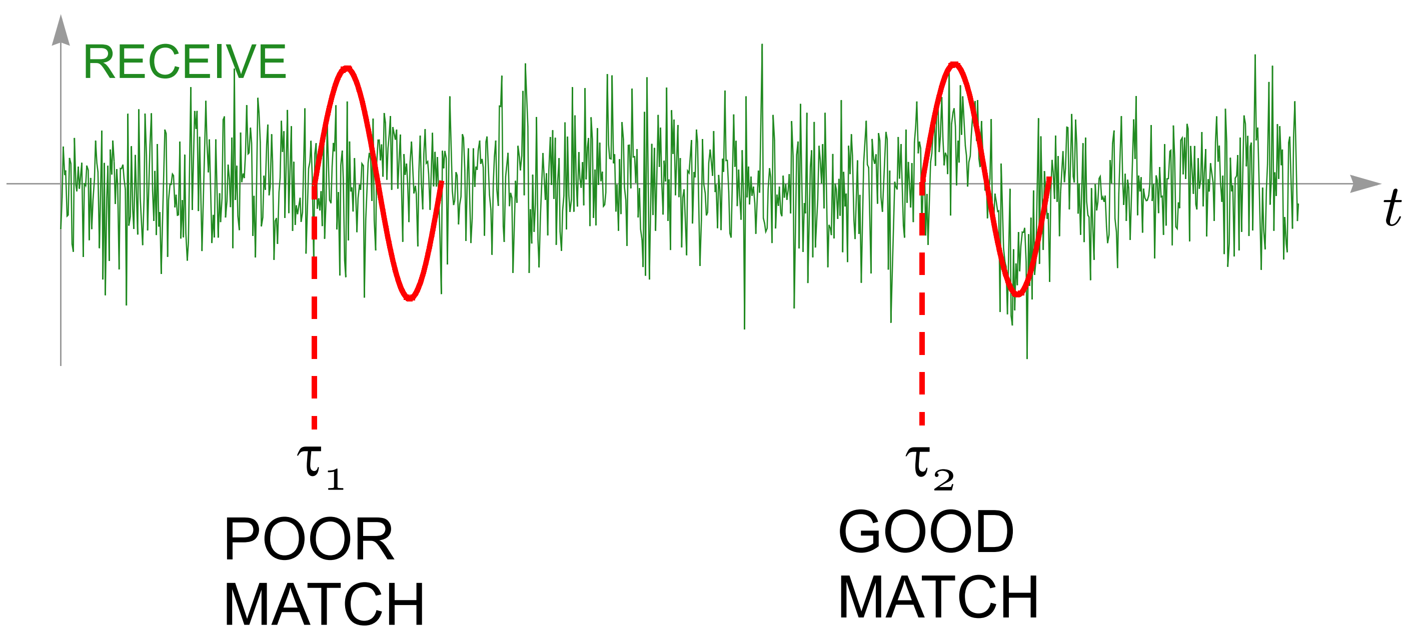


Key Problem: Estimate unknown delay T

Look for Best Match



Conceptually we want to *slide* template $s(t-\tau)$ and look for "best" match



How to Automate?

Convert to math: Choose τ to minimize:

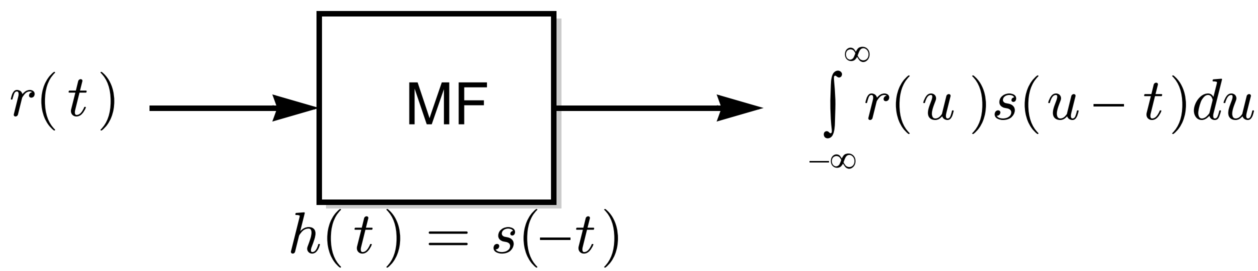
$$\int_{-\infty}^{\infty} (r(t) - s(t-\tau))^2 dt \quad = \quad E_r + E_s - 2\int_{-\infty}^{\infty} r(t)s(t-\tau)dt$$

Equivalently, choose τ to maximize

"crosscorrelation function" $R_{rs}(\tau) = \int_{-\infty}^{\infty} r(t)s(t-\tau)dt$

How to Implement Correlation: The Matched Filter

Recognize correlation integral as convolution of $r(t)$ with $h(t) = s(-t)$:



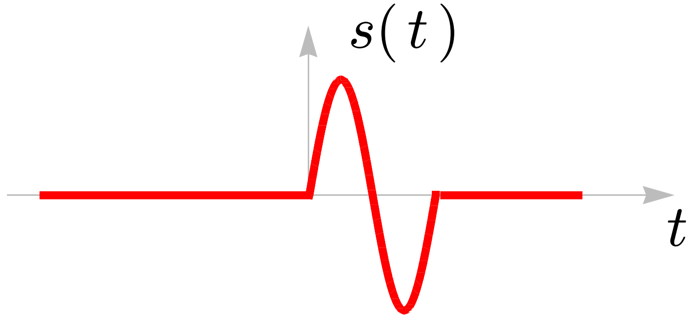
Verify using convolution!

The Matched Filter

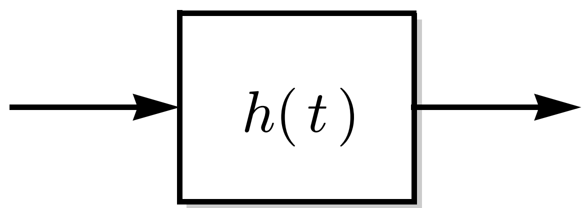
A filter whose impulse response is $h(t) = s^*(-t)$ is "matched" to $s(t)$.

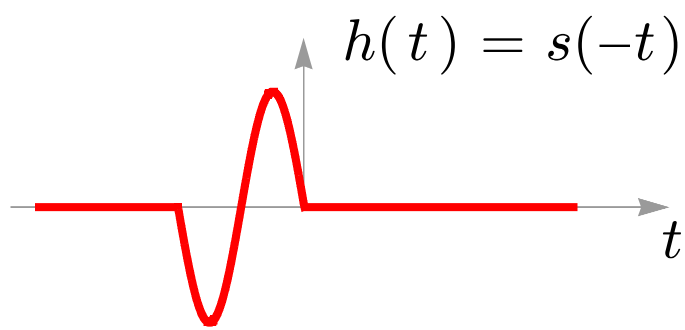
It "correlates" input with $s(t)$.

Example:



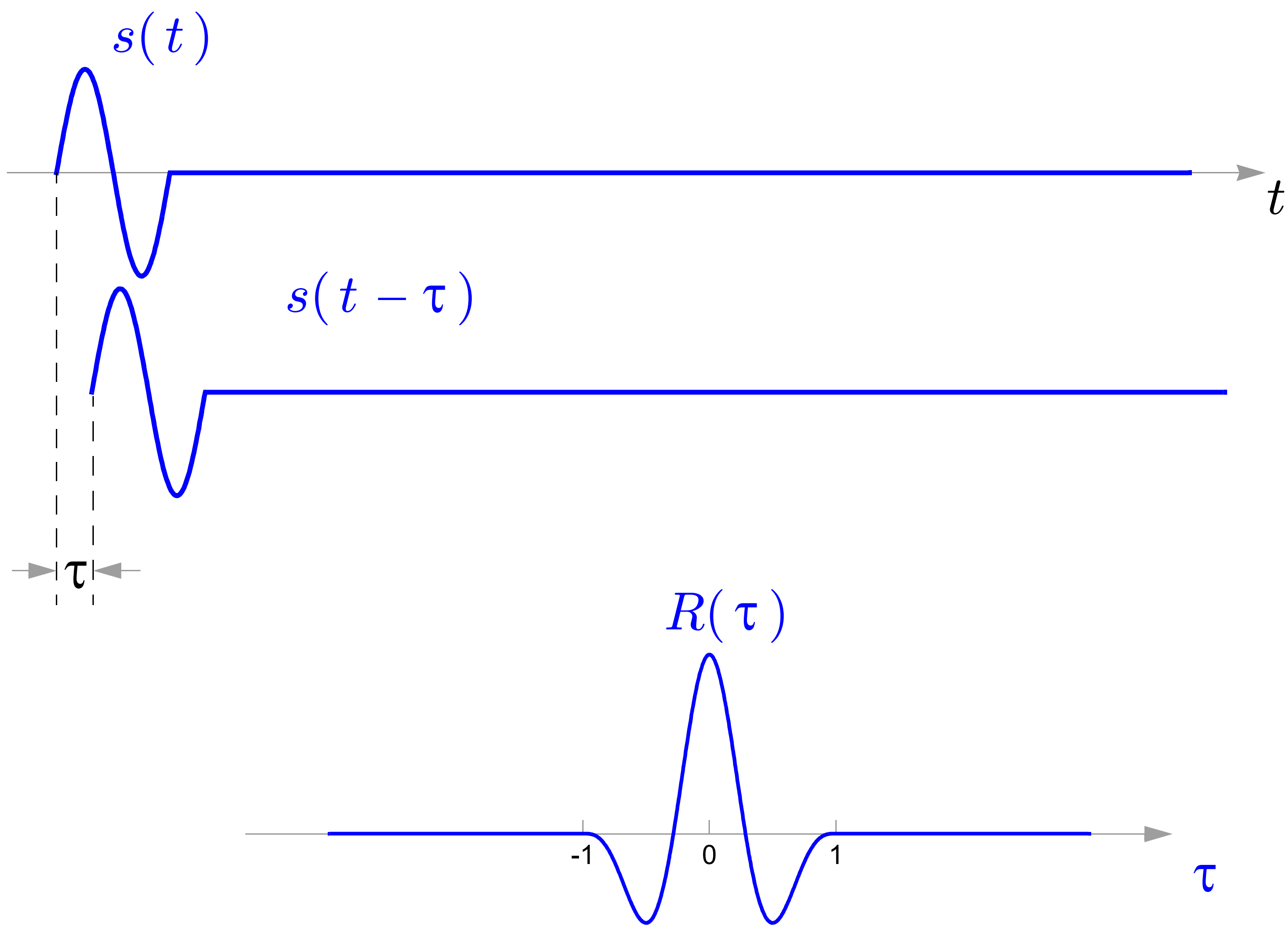
The filter below is "matched" to the above signal:





Autocorrelation Function

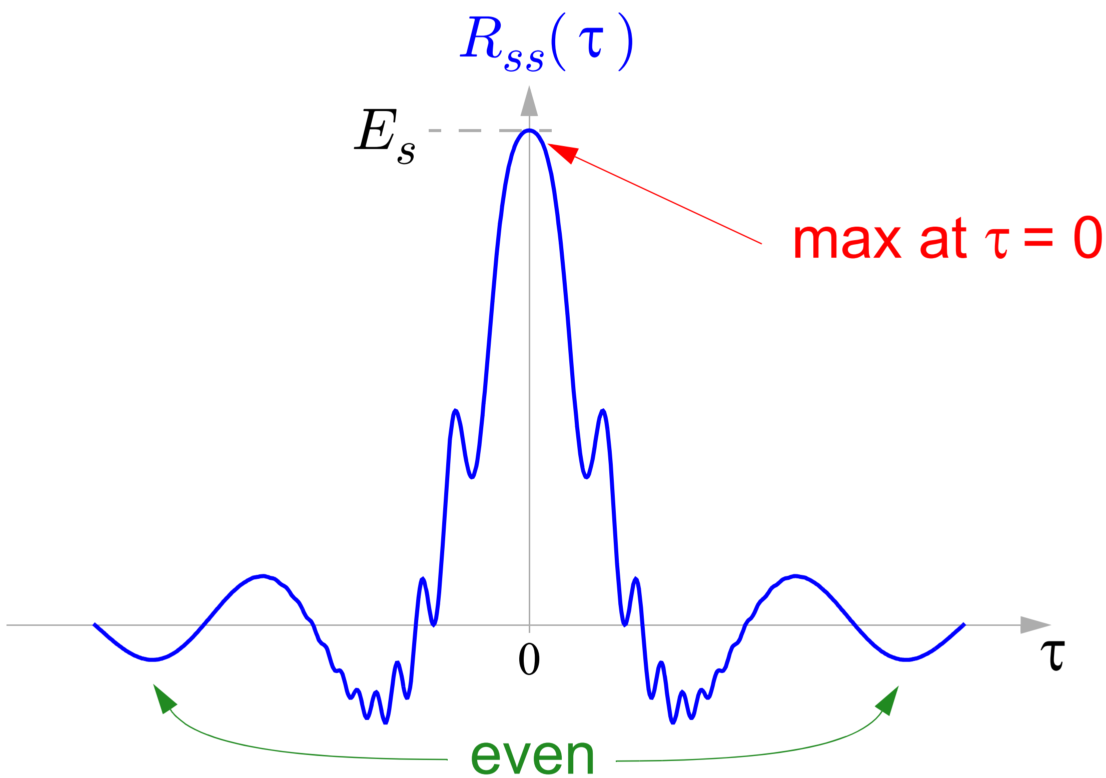
$$R_{ss}(\tau) = \int_{-\infty}^{\infty} s(t)s(t-\tau)dt$$



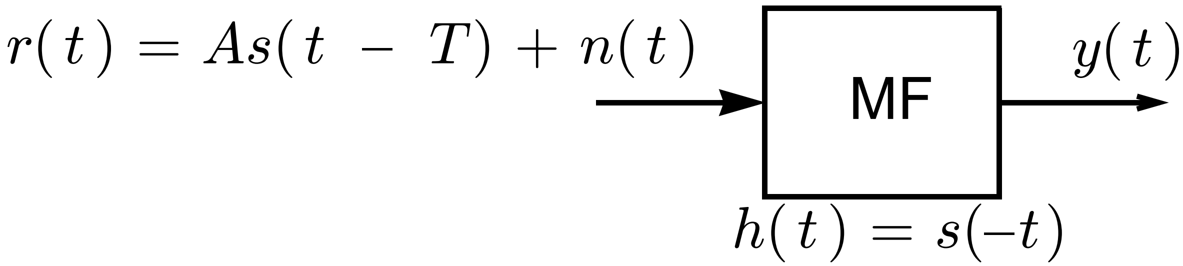
Properties of Autocorrelation

$$R_{ss}(\tau) = \int_{-\infty}^{\infty} s(t)s(t-\tau)dt$$

- $R_{ss}(0) = E_s$
- $R_{ss}(-\tau) = R_{ss}(\tau)$
- $|R_{ss}(\tau)| \leq R_{ss}(0)$



Back to Radar: Analyze MF Output



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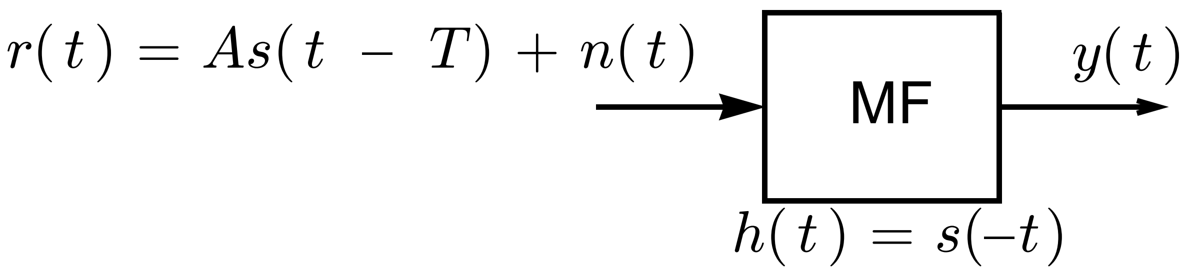
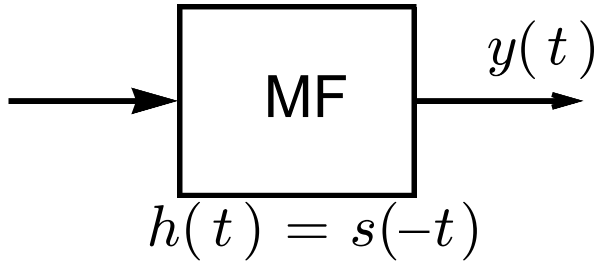


No noise: If $r(t) = As(t - T)$ then MF output is:

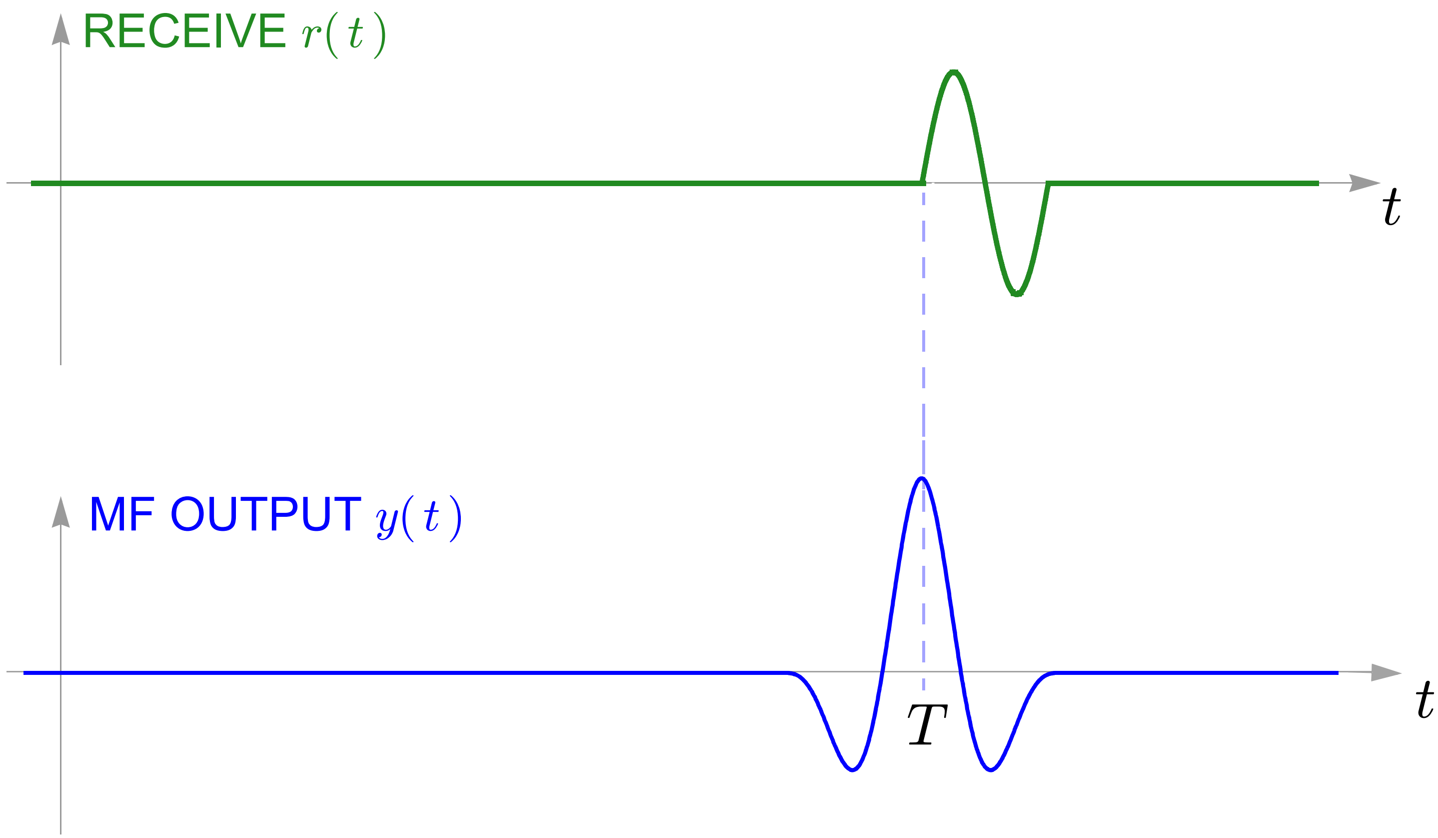
$$
\begin{aligned}
y(t) &= As(t-T) * h(t) \\
&= \int_{-\infty}^{\infty} As(\tau - T)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} As(\tau - T)s(\tau - t)d\tau \\
&= A\int_{-\infty}^{\infty} s(\tau)s(\tau - (t-T))d\tau \\
&= AR_{ss}(t-T)
\end{aligned}
$$

Without Noise

$r(t) = As(t - T)$

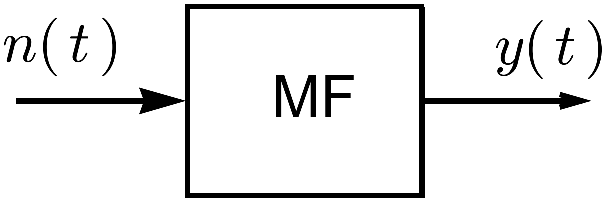


MF output is: $y(t) = AR_{ss}(t - T)$:



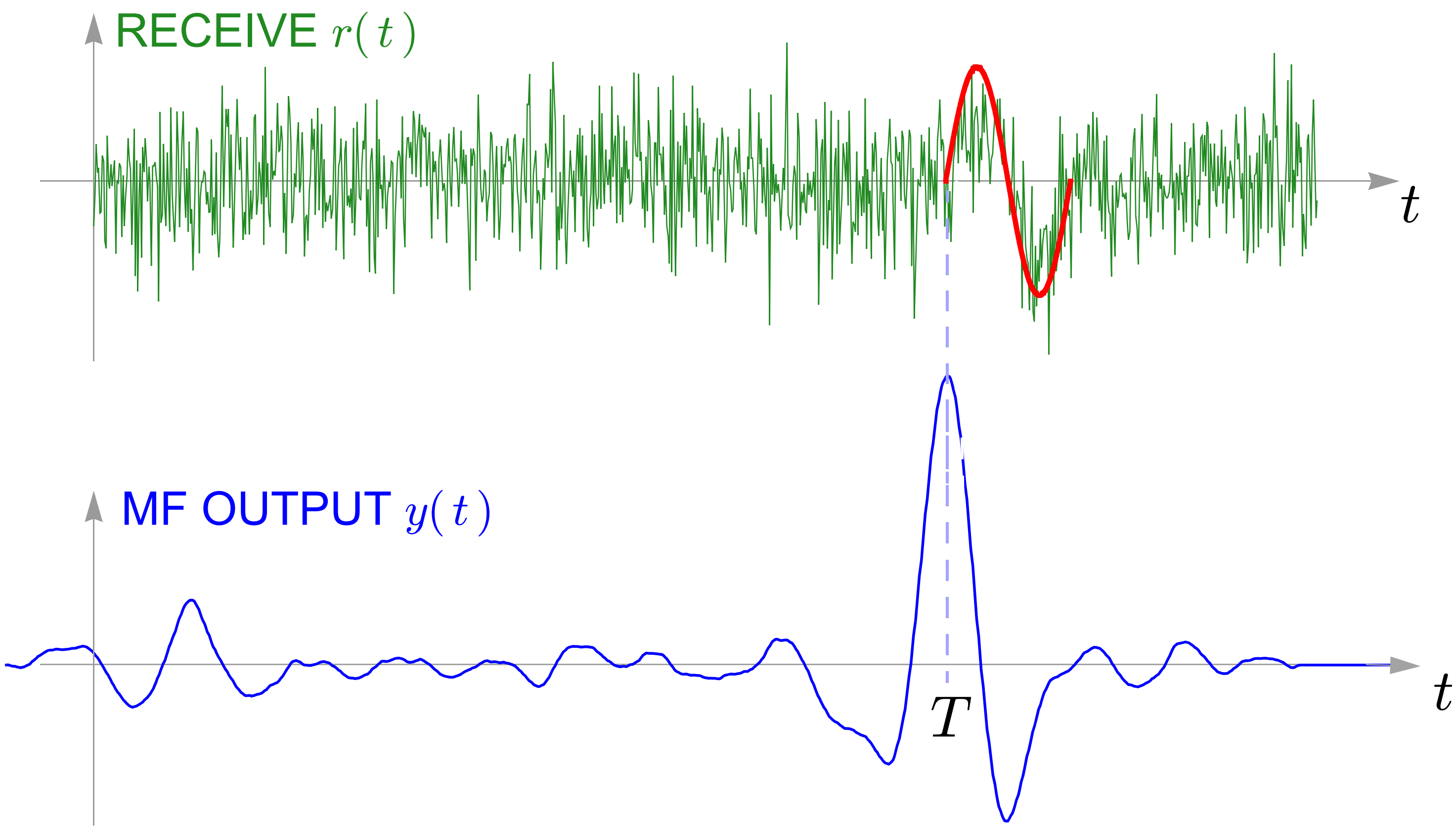
With Noise

$$r(t) = As(t - T) + n(t)$$

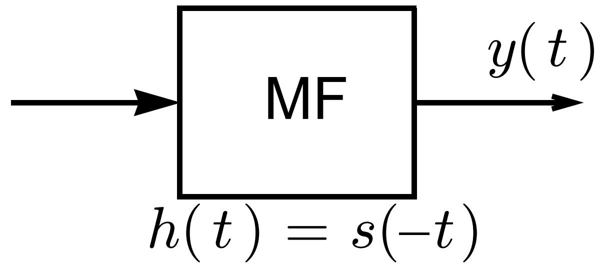


$$h(t) = s(-t)$$

MF output is: $y(t) = AR_{ss}(t - T) + \text{noise:}$



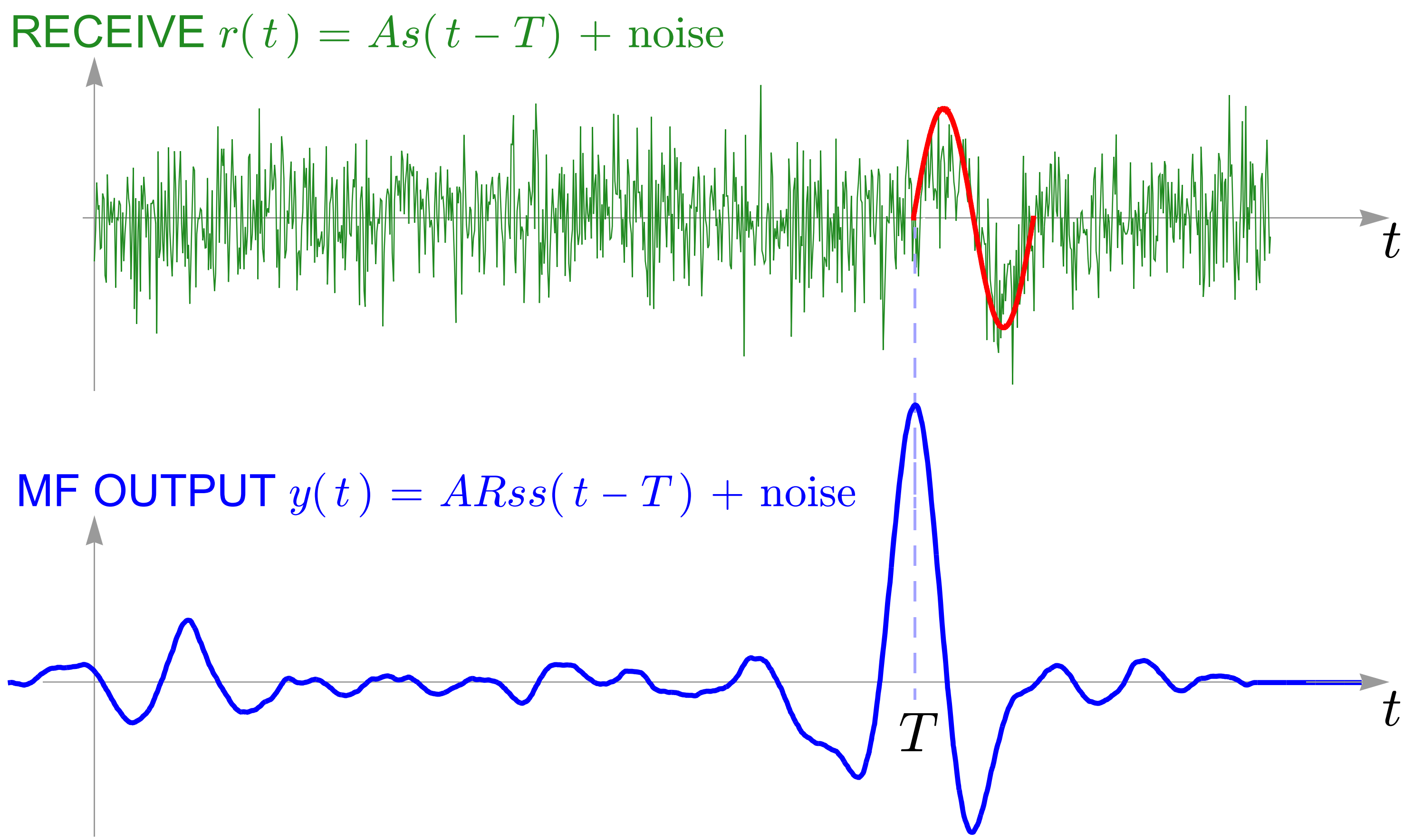
Two Common MF Inputs



#1: input is $s(t)$ $\Rightarrow$ output is $R_{ss}(t)$.

#2: input is $r(t) = As(t - T)$ $\Rightarrow$ output is $AR_{ss}(t - T)$.

With Noise



RECEIVE $r(t) = As(t-T) + \text{noise}$
t
MF OUTPUT $y(t) = ARss(t-T) + \text{noise}$
T
t

Good Signals for Radar

- Robust to noise $\Rightarrow$ Large peak in autocorrelation $\Rightarrow$ high energy E_s
- Good time resolution $\Rightarrow$ Narrow "thumbtack" autocorrelation peak
(Also helps distinguish multiple reflections)
  - ▹ In theory, can achieve by transmitting $s(t) = \delta(t)$: not practical!
  - ▹ Can also achieve via *pulse compression:*

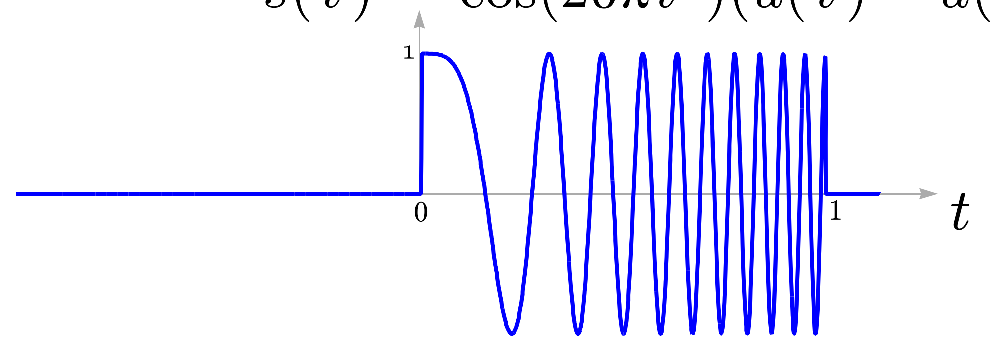
    looong-duration $s(t)$ whose autocorrelation compresses to a thumbtack

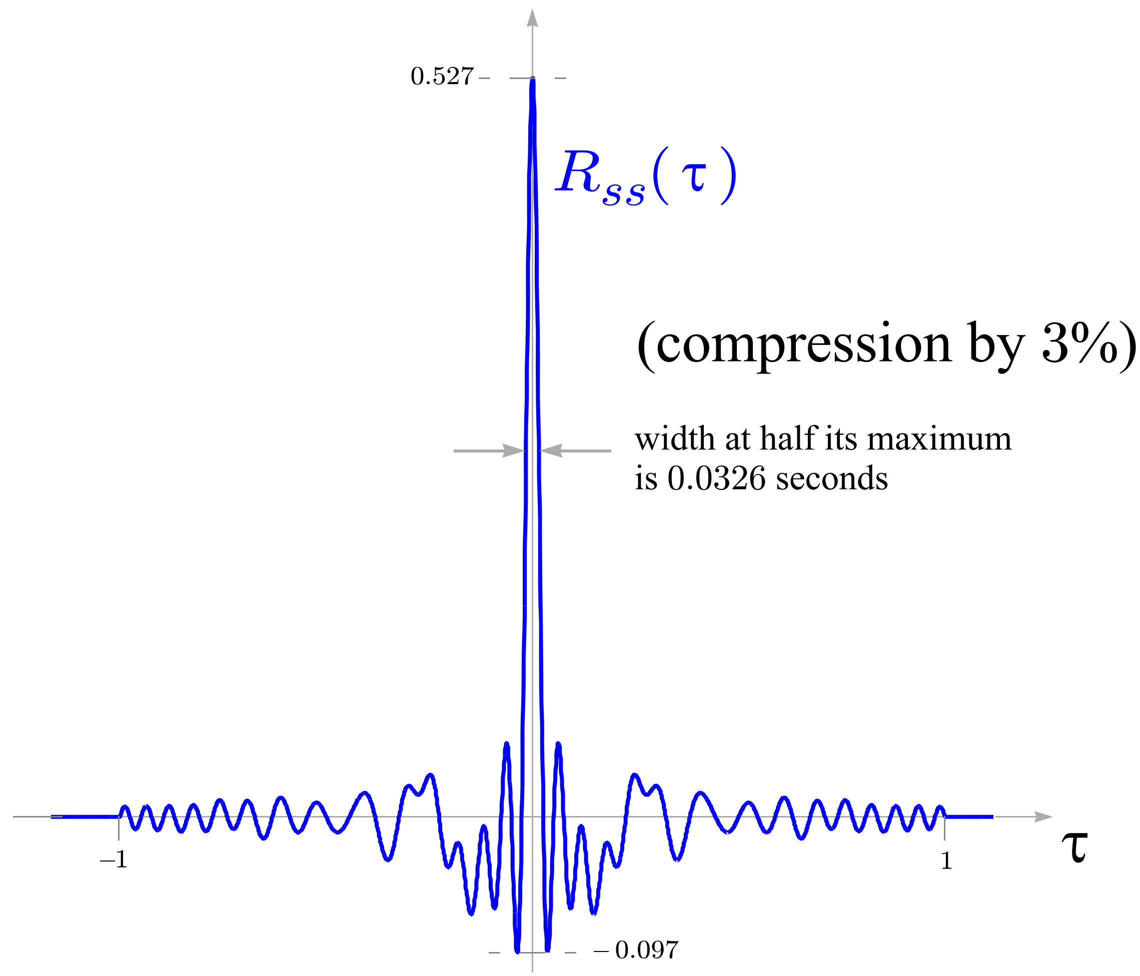
Pulse compression examples:

- chirp (LFM)
- pseudonoise
- binary codes

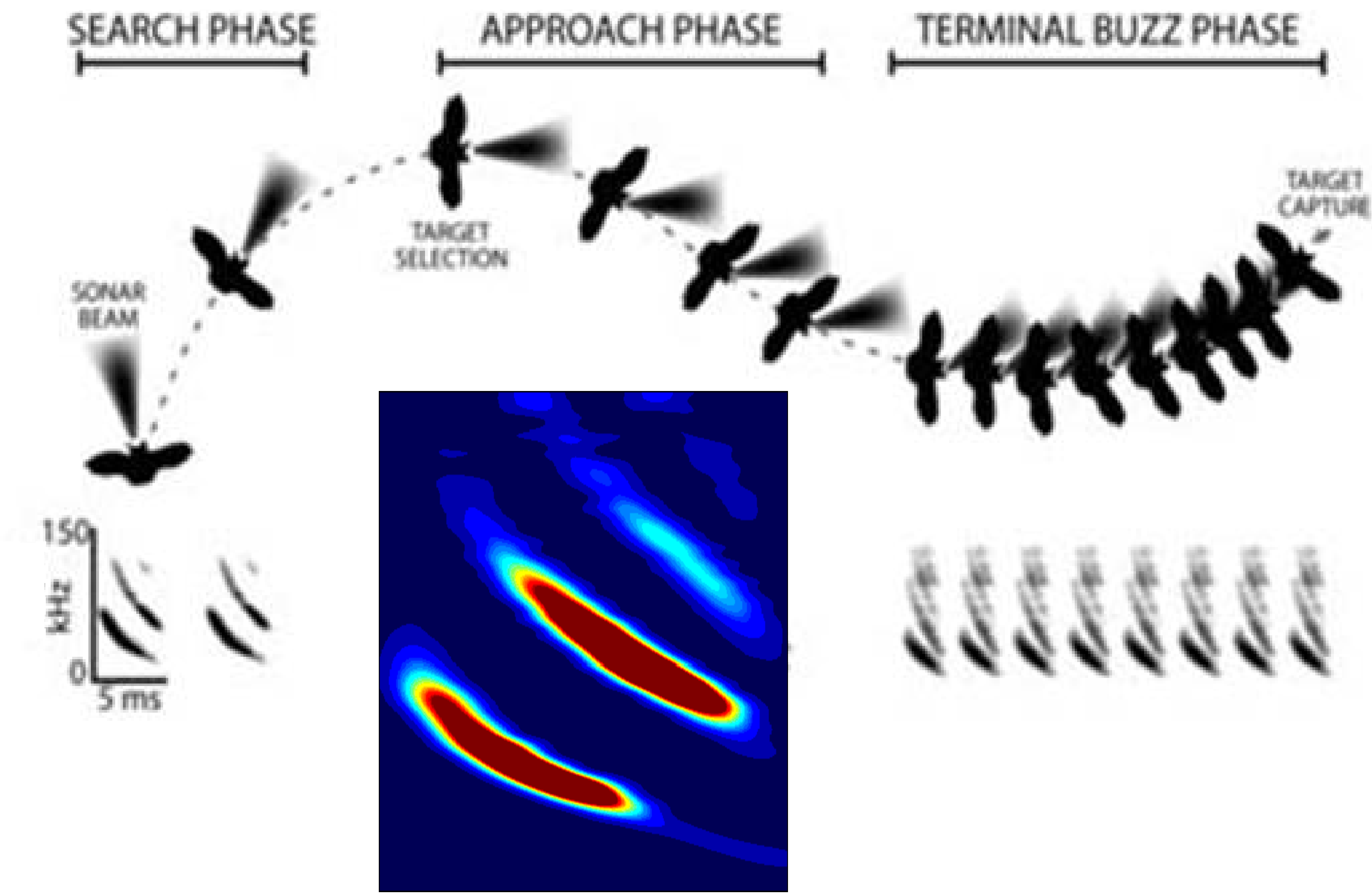
Autocorrelation of Chirp

$$s(t) = \cos(20\pi t^2)(u(t) - u(t-1))$$

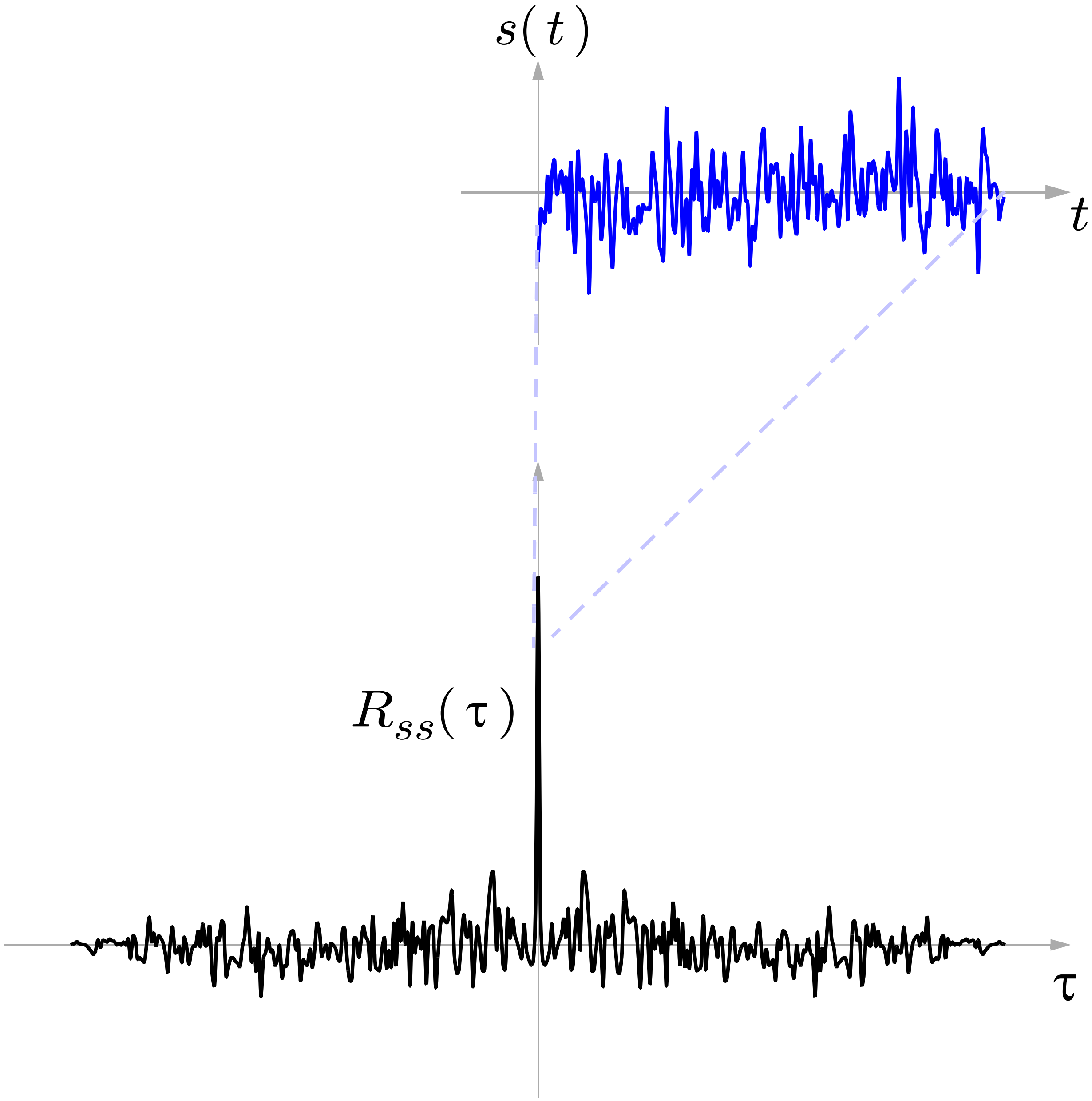




Bat Spectrograms

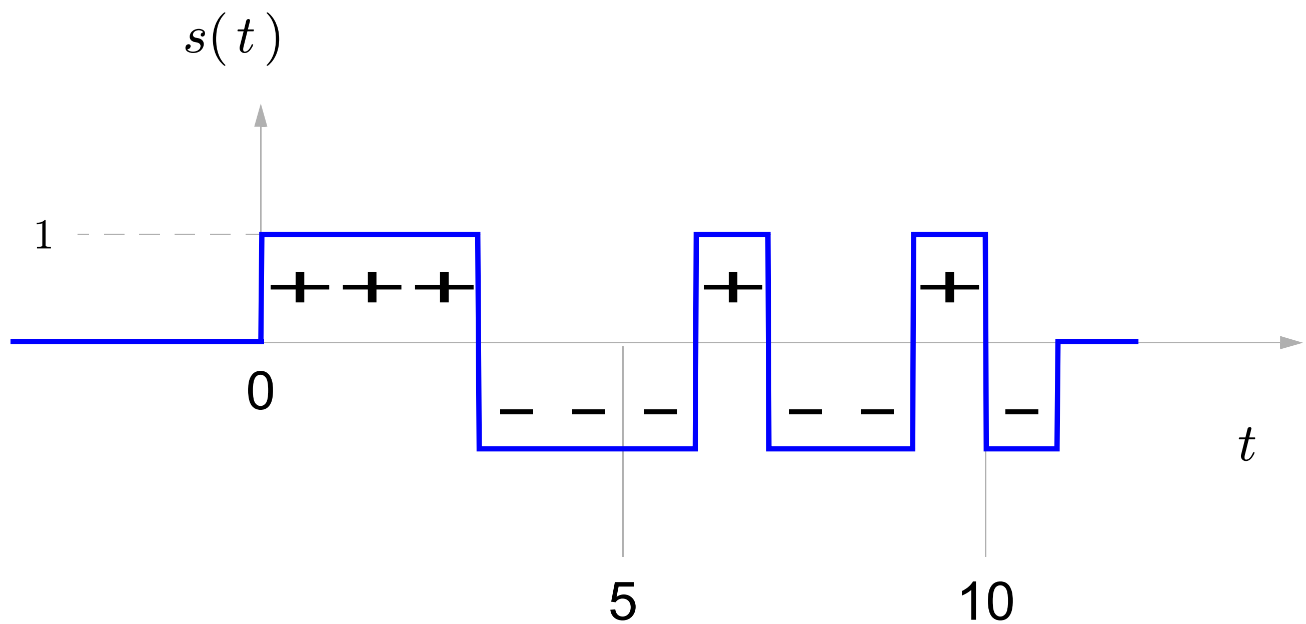


PseudoNoise



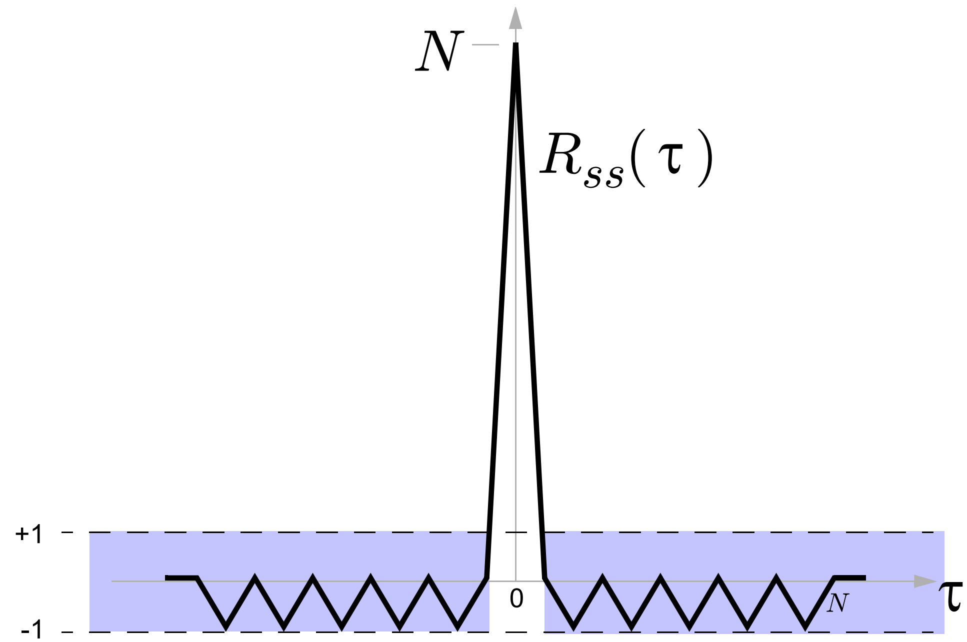
Binary Code Pulse Train

Length-11 Barker code (**11100010010**):



Barker Codes

$R(0) = N$, $|R(\tau)| \le 1$ for $|\tau| \ge 1$



tails confined here

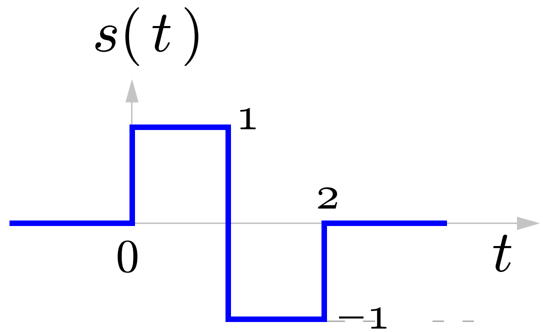
Known Barker codes for $N \in \{2, 3, 4, 5, 7, 11, 13\}$:

$N = 2$:	+ -
$N = 3$:	+ + -
$N = 4$:	+ + - +
$N = 5$:	+ + + - +
$N = 7$:	+ + + - - + -
$N = 11$:	+ + + - - - + - - + -
$N = 13$:	+ + + + + - - + + - + - +

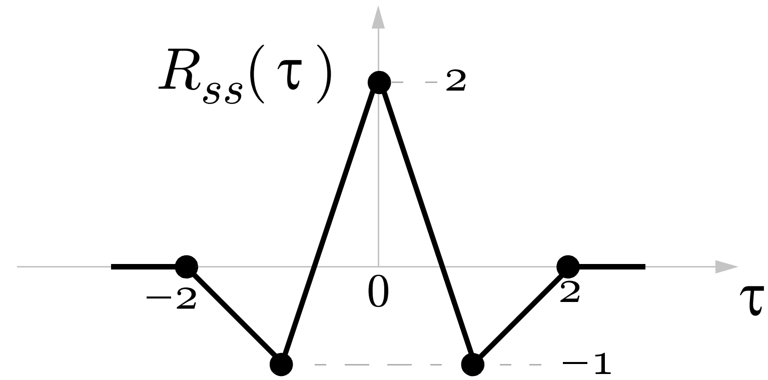
(CHIRP-LIKE?)

Conjecture: There are no others!

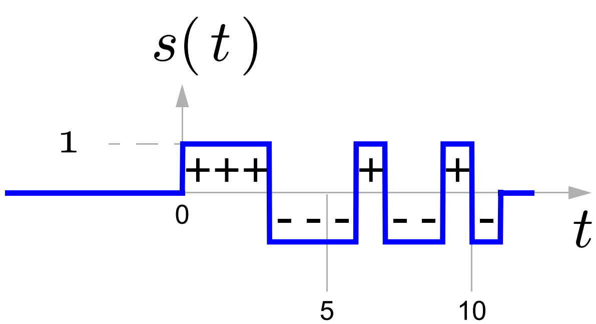
Binary Code Autocorrelation Examples



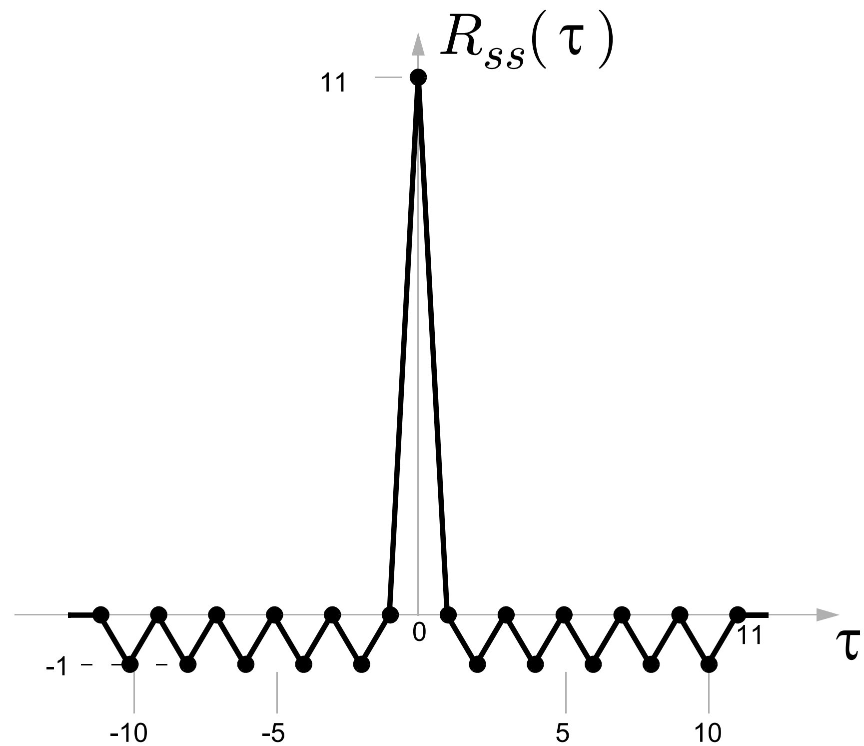
⇒



Length-11 Barker code:
+++---+--+-



⇒



Shortcut:
Convolving Piecewise Constant Signals

- Compute convolution **only <u>at</u> boundaries** between time regions
- For autocorrelation function, exploit even property!
- Connect the dots using straight lines!