Lecture 23: Tue Nov 3, 2020

Reminder:

- Homework 10 due Thursday

Lecture

- inverse Laplace via PFE
- Translating between equivalent representations:
    - ▹ pole-zero plot
    - ▹ impulse response
    - ▹ frequency response
    - ▹ transfer function

Quiz 2
in 9 Days

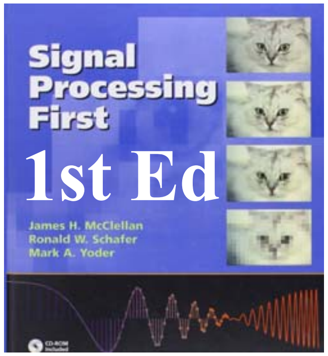
Coverage:

- lectures 10 – 23
- homeworks 6 – 10
- Chapters 7 – 10
  of 3084 PDF book
- Chapters 4, 11 & 12
  of 2026 SP1st (1st Ed)
- Youtube module 18 (lec 15)
  through module 33 (lec 31)

:10	Thu Sep 17	More Fourier series, intro to Fourier transform
:11	Tue Sep 22	The Fourier Transform: famous pairs
:12	Thu Sep 24	Fourier Transform properties
:13	Tue Sep 29	Fourier transform: using the tables
:14	Thu Oct 1	QUIZ 1
:15	Tue Oct 6	MF in the frequency domain; Parseval's relation
:16	Thu Oct 8	Time-bandwidth product, the Gaussian pulse, AM modulation and demod
:17	Tue Oct 13	AM demodulation, downconversion, I&Q
:18	Thu Oct 15	up and downconversion, complex envelopes, I & Q, envelope & phase
:19	Tue Oct 20	Review I & Q; downconversion in the frequency domain
:20	Thu Oct 22	The sampling theorem revisited; intro to Laplace
:21	Tue Oct 27	The Laplace transform
:22	Thu Oct 29	Laplace: Tables 1 and 2
:23	Tue Nov 3	Translating between pole/zero, H(s), h(t); inverse Laplace; PFE

Coverage from 3084 PDF

Coverage from SP First



4 Sampling and Aliasing 71

11 Continuous-Time Fourier Transform 307

12 Filtering, Modulation, and Sampling 346

Coverage from Youtube

Modules 18 - 33:

Handouts for ECE 3084

- syllabus
- 3084 book

Supplemental Material

- Step and Delta functions (from https://math.mit.edu/~stoopn/18.031/)
- Summer 2026 ECE3084 video lecture playlist (youtube)
- ECE3084 Resource Page: Videos and lecture notes.
- MIT Class: Videos, lecture notes, and problems.

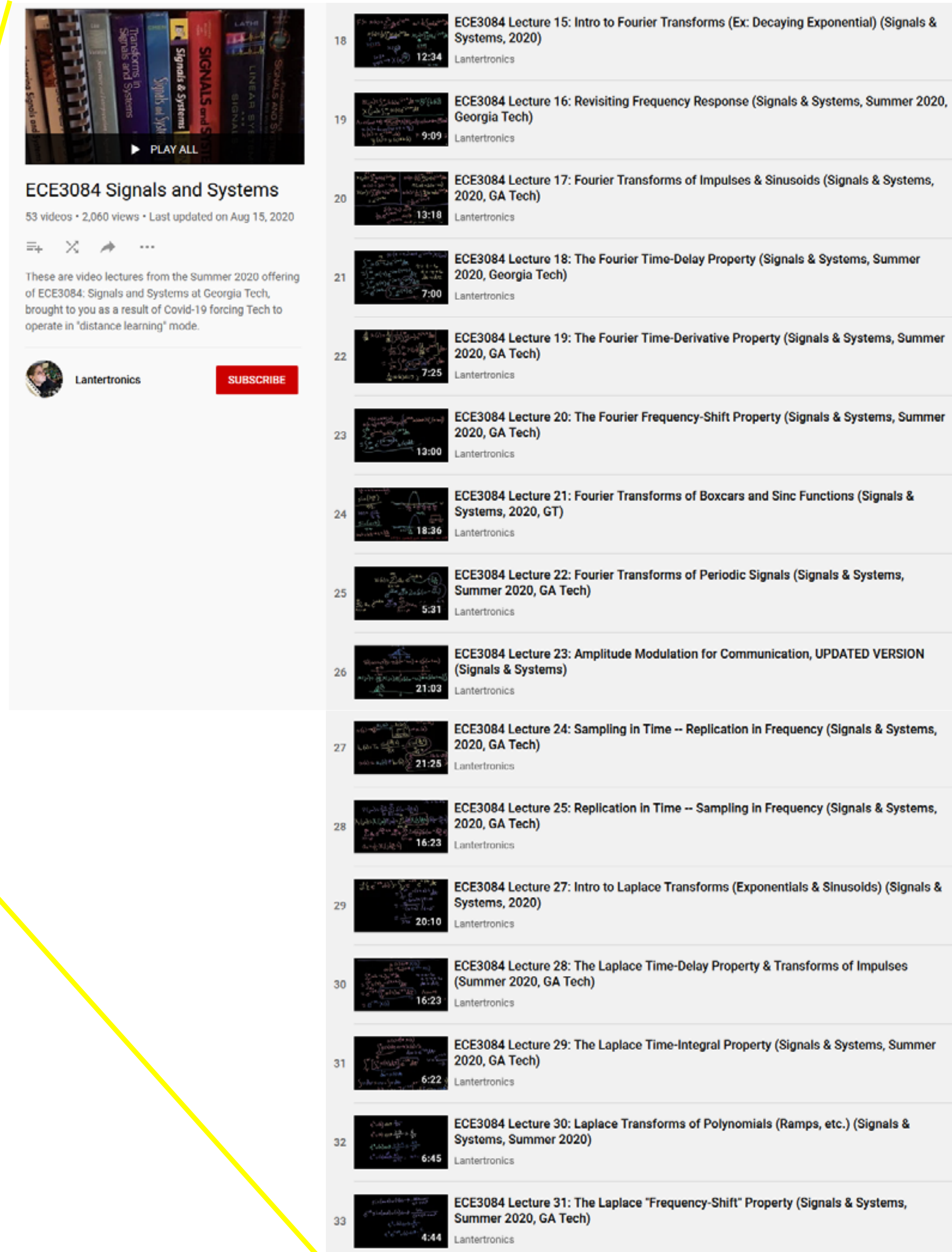


Table 1: Pairs

Impulse	$\delta(t)$	1	✓
Delayed Impulse	$\delta(t-t_0), \quad t_0 \geq 0$	e^{-st_0}	
Step	$u(t)$	$\frac{1}{s}$	✓
Rectangular Pulse	$u(t)-u(t-T), \quad T>0$	$\frac{1-e^{-sT}}{s}$	✓
Ramp	$tu(t)$	$\frac{1}{s^2}$	✓
Polynomial	$t^k u(t), \quad k \geq 0$	$\frac{k!}{s^{k+1}}$	✓
Exponential	$e^{-at}u(t)$	$\frac{1}{s+a}$	✓
Polynomial $\times$ Exponential	$t^k e^{-at}u(t)$	$\frac{k!}{(s+a)^{k+1}}$	✓
Cosine	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	✓
Sine	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	✓
Exponential $\times$ Cosine	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	
Exponential $\times$ Sine	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	

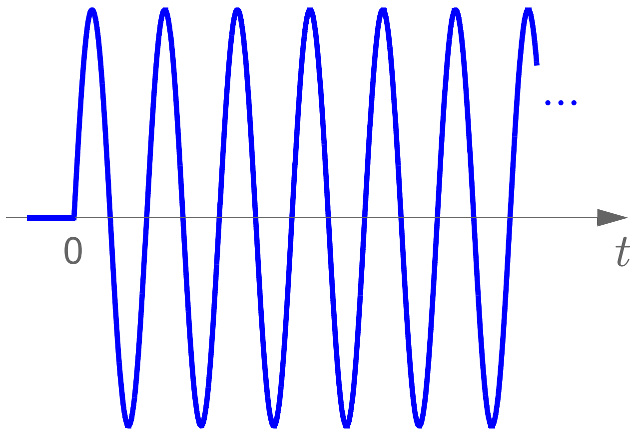
Table 2: Properties

Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$ ✓
Right-Shift	$x(t - t_0), \quad t_0 \geq 0$	$e^{-st_0} X(s)$ ✓
Time Scaling	$x(at), \quad a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$
First Derivative	$\dot{x}(t) = \frac{d}{dt} x(t)$	$sX(s) - x(0)$
Second Derivative	$\ddot{x}(t) = \frac{d^2}{dt^2} x(t)$	$s^2 X(s) - sx(0) - \dot{x}(0)$
Integration	$\int_0^t x(\tau) d\tau$	$\frac{X(s)}{s}$ ✓
Modulation	$x(t)e^{at}$	$X(s - a)$ ✓
Convolution	$x(t) * h(t)$	$X(s)H(s)$ ✓
Final Value Theorem	$\lim_{t \to \infty} x(t)$, (if limit exists)	$\lim_{s \to 0} sX(s)$
Initial Value Theorem	$\lim_{t \to 0} x(t)$, (if limit exists)	$\lim_{s \to \infty} sX(s)$

s-domain differentiation $\quad -tx(t) \quad \frac{d}{ds} X(s)$ ✓

Pop Quiz

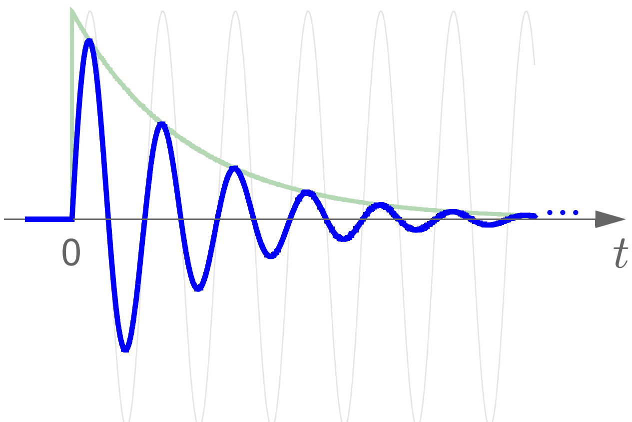
$$h(t) = \sin(t)\,u(t) \qquad \longleftrightarrow \qquad \text{Pole-zero plot?}$$



Pop Quiz

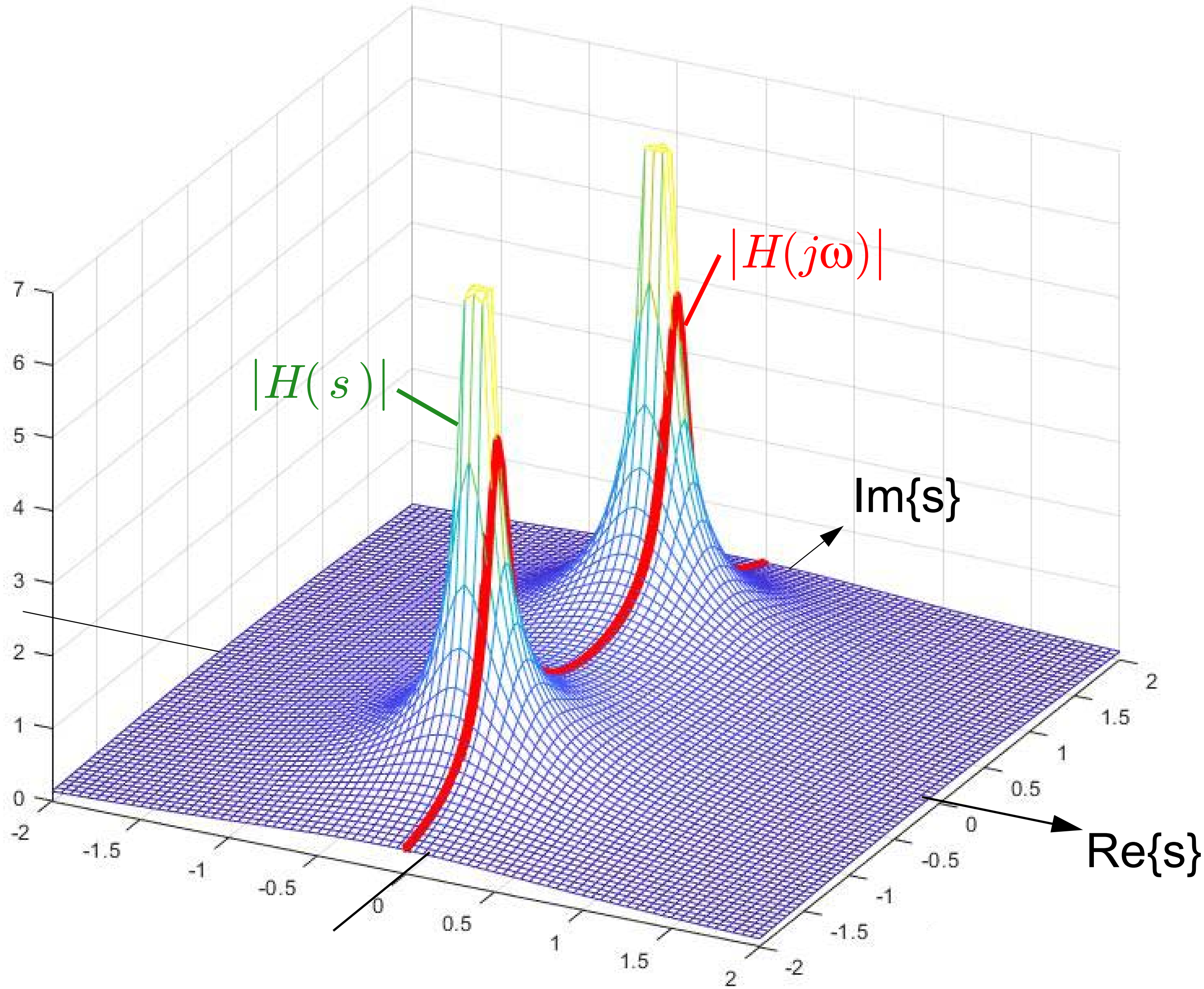
$$h(t) = e^{-0.1t}\sin(t)\,u(t) \quad \longleftrightarrow$$

Pole-zero plot?



$H(s)$?

Magnitude response $|H(j\omega)|$?



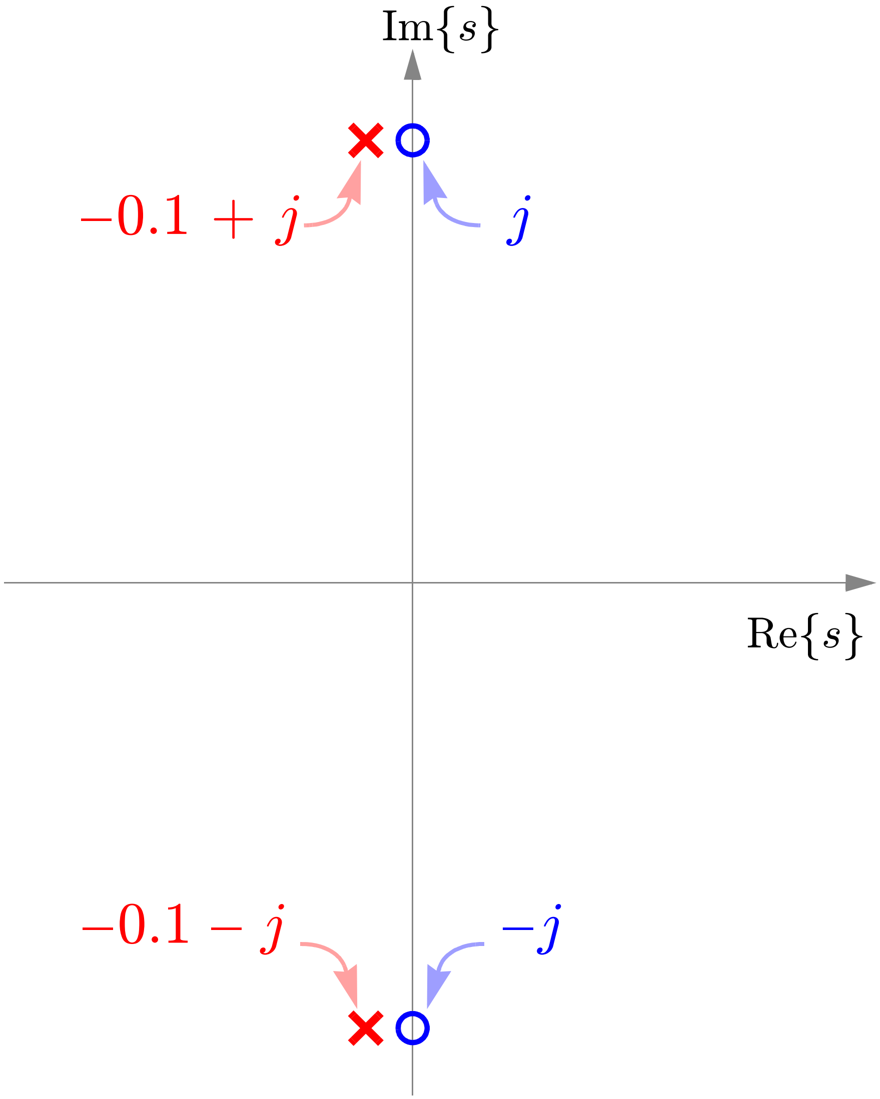
$|H(s)|$
$|H(j\omega)|$
Im{s}
Re{s}
7
6
5
4
3
2
1
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

What Kind of Filter is This?

$$H(s) = \frac{s^2 + 1}{s^2 + 0.2s + 1.01}$$

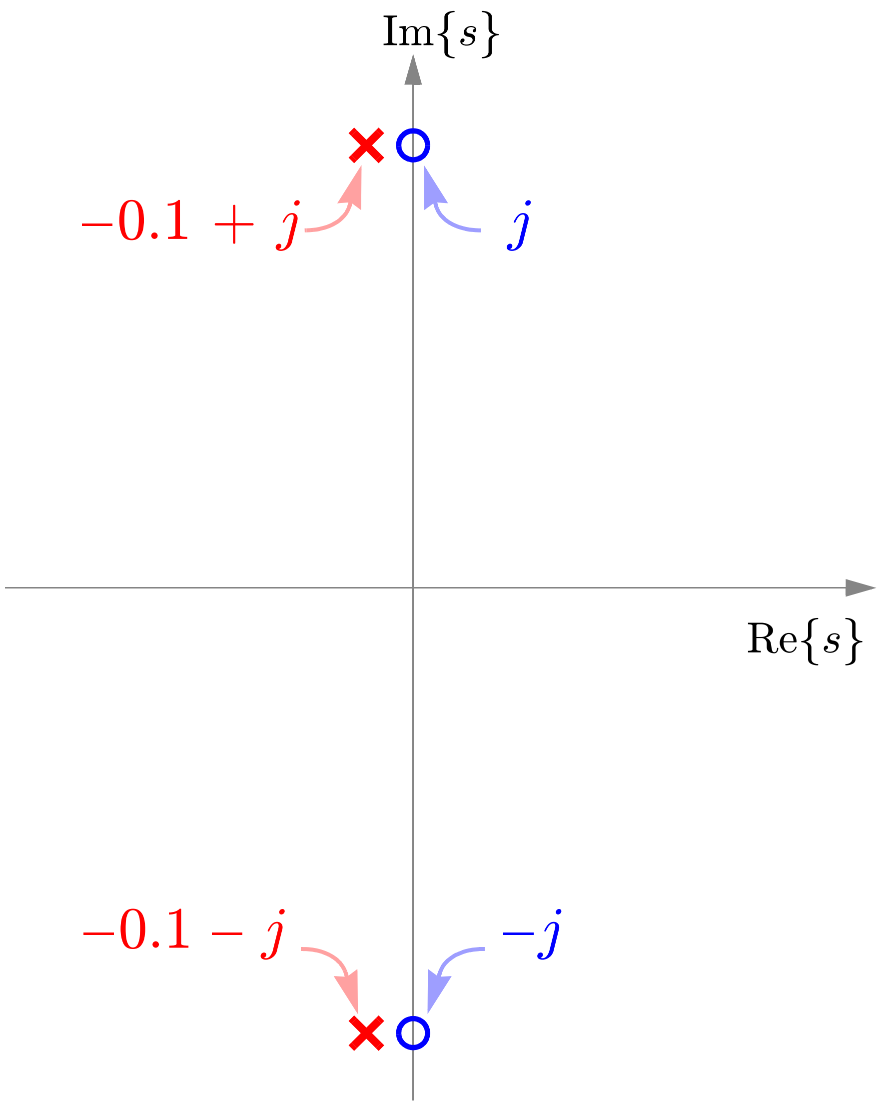
What Kind of Filter is This?

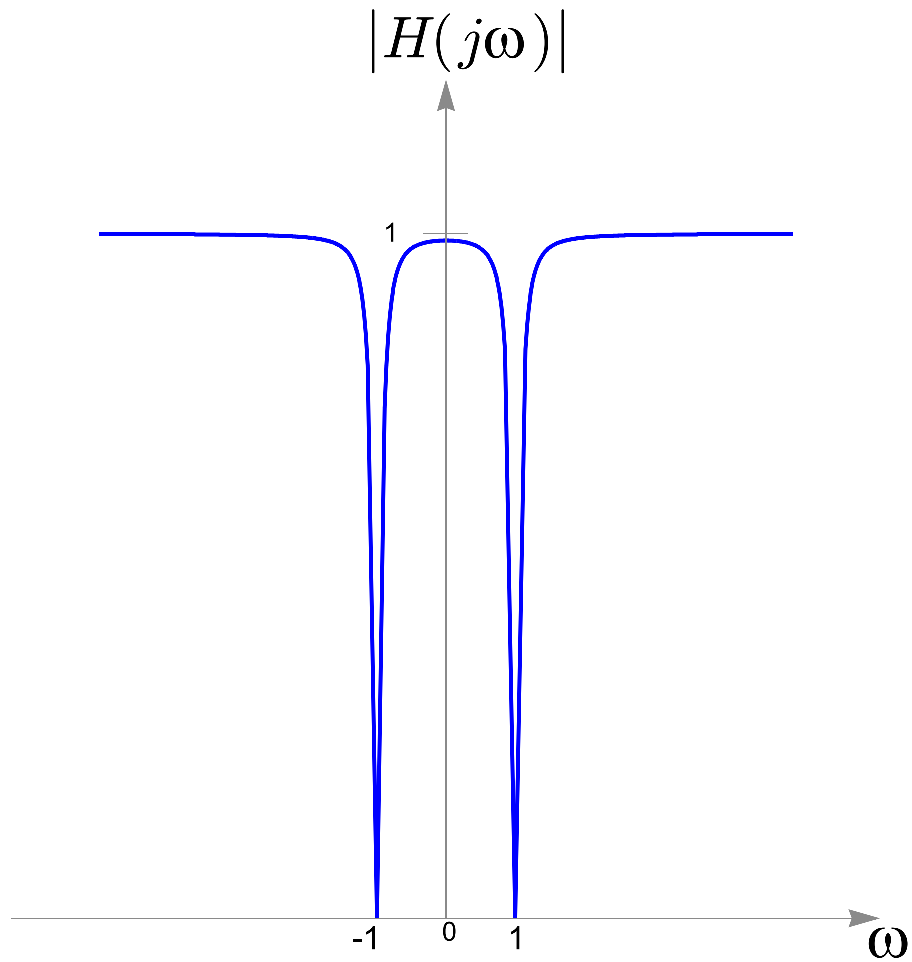
$$H(s) = \frac{s^2 + 1}{s^2 + 0.2s + 1.01}$$



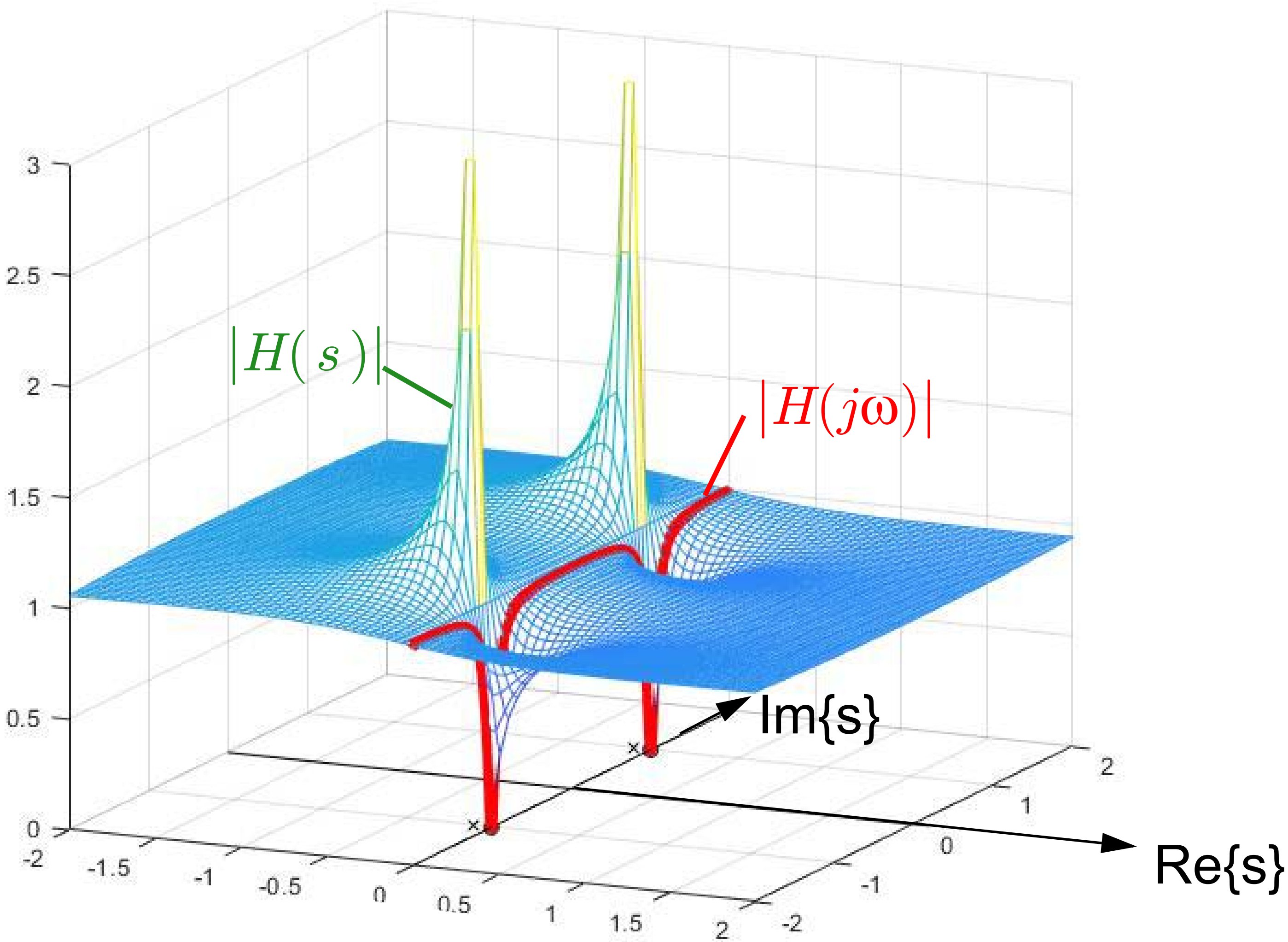
What Kind of Filter is This?

$$H(s) = \frac{s^2 + 1}{s^2 + 0.2s + 1.01}$$



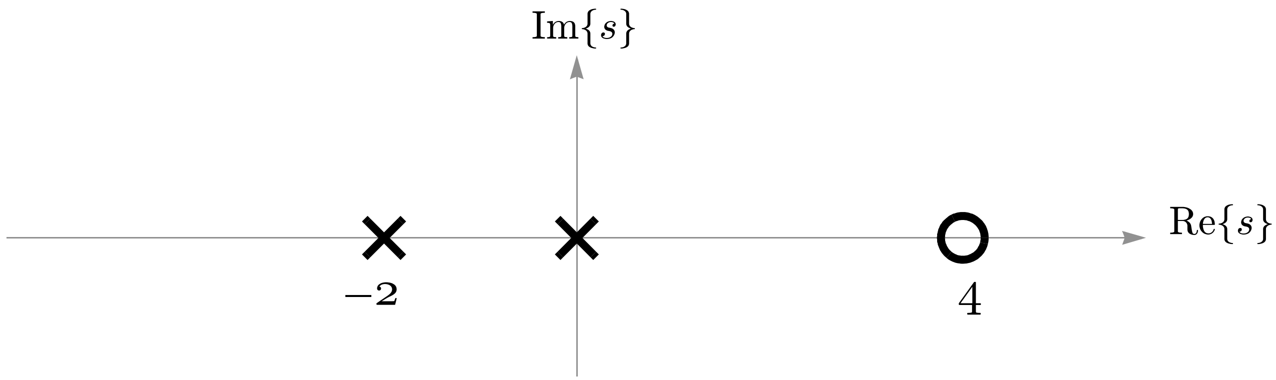


$\Rightarrow$ “notch” filter



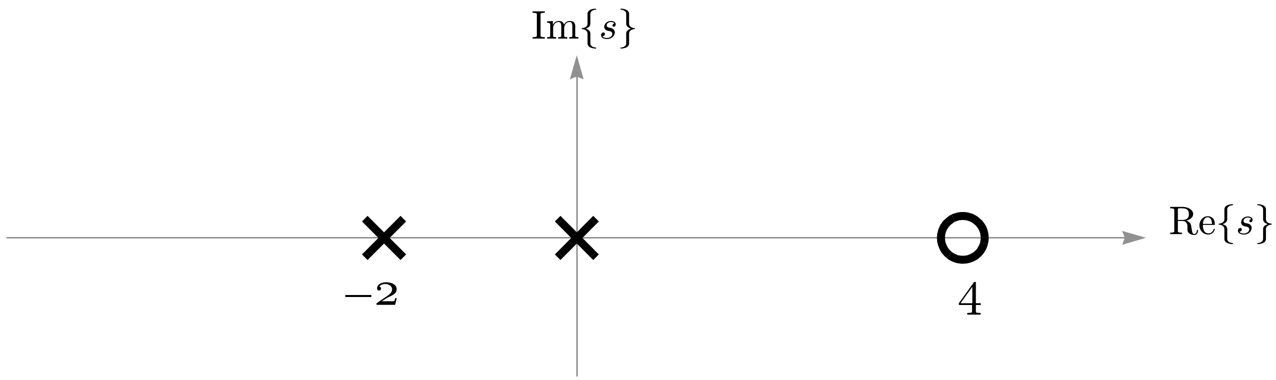
$|H(s)|$
$|H(j\omega)|$
Im{s}
Re{s}

From Pole-Zero Plot to $h(t)$



Find $h(t)$.

From Pole-Zero Plot to $h(t)$



Find $h(t)$.

$H(s) = \dfrac{s-4}{s(s+2)}$ ... How to take inverse Laplace transform?

Inverse Laplace Transform

Contour integral not needed for rational $X(s) = \dfrac{P(s)}{Q(s)}$;

Instead, use PFE and tables.

Assume order$P(s) < N =$ order$Q(s)$.

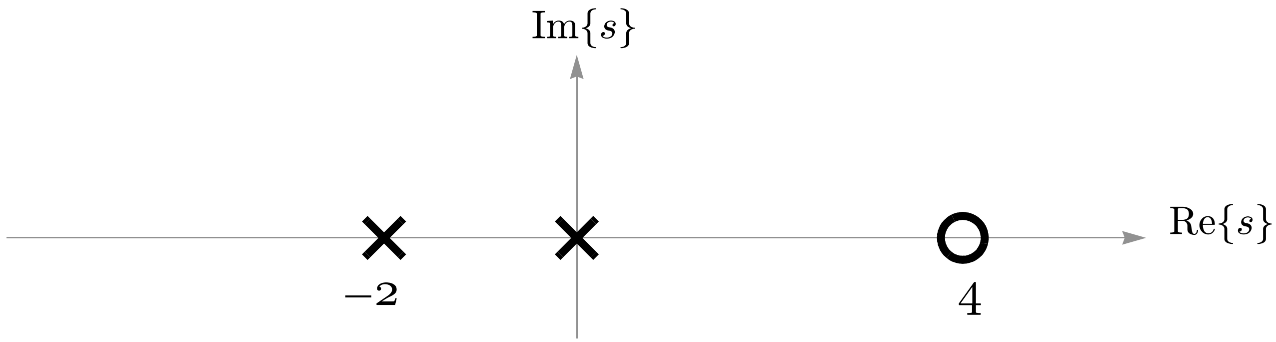
(If not, do polynomial division and apply PFE only to remainder.)

Case 1: All poles distinct:

$$X(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \ldots + + \frac{A_N}{s-p_N}$$

Use "residue" or "clear denom" strategy to find coefficients.

From Pole-Zero Plot to $h(t)$



Find $h(t)$.

$$H(s) = \frac{s-4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$ PARTIAL FRACTION EXPANSION:

- clear denominator $\Rightarrow s - 4 = A(s+2) + Bs$
- equate powers of $s^0 \Rightarrow A = -2$
- equate powers of $s^1 \Rightarrow A + B = 1 \Rightarrow B = 3$
- Laplace pairs table $\Rightarrow h(t) = -2u(t) + 3e^{-2t}u(t)$.

```
>> [r, p] = residue([1 -4],[1 2 0])

r =

     3
    -2


p =

    -2
     0
```

Another Residue Example

$\text{order}_{\text{num}} < \text{order}_{\text{den}}$, no repeated roots:

$$X(s) = \frac{(s+5)(s+6)}{(s+1)(s+2)(s+3)(s+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

where:

$$A = (s+1)X(s)\Big|_{s=-1} = \frac{(s+5)(s+6)}{\cancel{(s+1)}(s+2)(s+3)(s+4)}\Big|_{s=-1} = \frac{20}{6}$$

$$B = (s+2)X(s)\Big|_{s=-2} = \frac{(s+5)(s+6)}{(s+1)\cancel{(s+2)}(s+3)(s+4)}\Big|_{s=-2} = -6$$

$$C = (s+3)X(s)\Big|_{s=-3} = \frac{(s+5)(s+6)}{(s+1)(s+2)\cancel{(s+3)}(s+4)}\Big|_{s=-3} = 3$$

$$D = (s+5)X(s)\Big|_{s=-5} = \frac{(s+5)(s+6)}{(s+1)(s+2)(s+3)\cancel{(s+4)}}\Big|_{s=-4} = \frac{-1}{3}$$

Pop Quiz

Find $x(t)$ when $$X(s) = \frac{1}{s^2 + s + 1}$$

Pop Quiz

```
>> roots([1 1 1])

ans =

  -0.5 + 0.8660i
  -0.5 - 0.8660i
```

Find $x(t)$ when
$$X(s) = \frac{1}{s^2 + s + 1}$$

$$= \frac{1}{(s + 0.5 - \frac{\sqrt{3}}{2}j)(s + 0.5 + \frac{\sqrt{3}}{2}j)}$$

$$= \frac{\frac{-j}{\sqrt{3}}}{s + 0.5 - \frac{\sqrt{3}}{2}j} + \frac{\frac{j}{\sqrt{3}}}{s + 0.5 + \frac{\sqrt{3}}{2}j}$$ **(PFE)**

From table

$$\Rightarrow x(t) = \ldots$$

There is an easier way!

There is an Easier Way!

1. Complete the square:

$$\Rightarrow \; X(s) = \frac{1}{s^2 + s + 1} = \frac{1}{(s+0.5)^2 + 0.75}.$$

2. Use modulation property.
   (Or just use table 2, it's already there!)

Either way, rewrite $X(s) = \dfrac{1}{(s+0.5)^2 + 0.75} = \dfrac{1}{\omega_0} \dfrac{\omega_0}{(s+0.5)^2 + \omega_0^2}$

$$\Rightarrow x(t) = \frac{1}{\omega_0} e^{-0.5t} \cos(\omega_0 t) u(t)$$

$$= \frac{2}{\sqrt{3}} e^{-0.5t} \cos(\frac{\sqrt{3}}{2} t) u(t)$$