

Lecture 22: Thu Oct 29, 2020

Reminder:

- Homework 9 due tonight

Lecture

- Laplace transforms: finish table 1 and table 2
- poles and zeros

3084 Reading

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About Laplace: $X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$

It's better than Fourier when:

- ▷ signals might not have Fourier transforms
- ▷ analyzing systems that may or may not be unstable
- ▷ solving diff. eq's

When causal: setting $s = j\omega$ reduces LT to FT.

(This explains why we bother to carry the “ j ” around in $X(j\omega)$!)

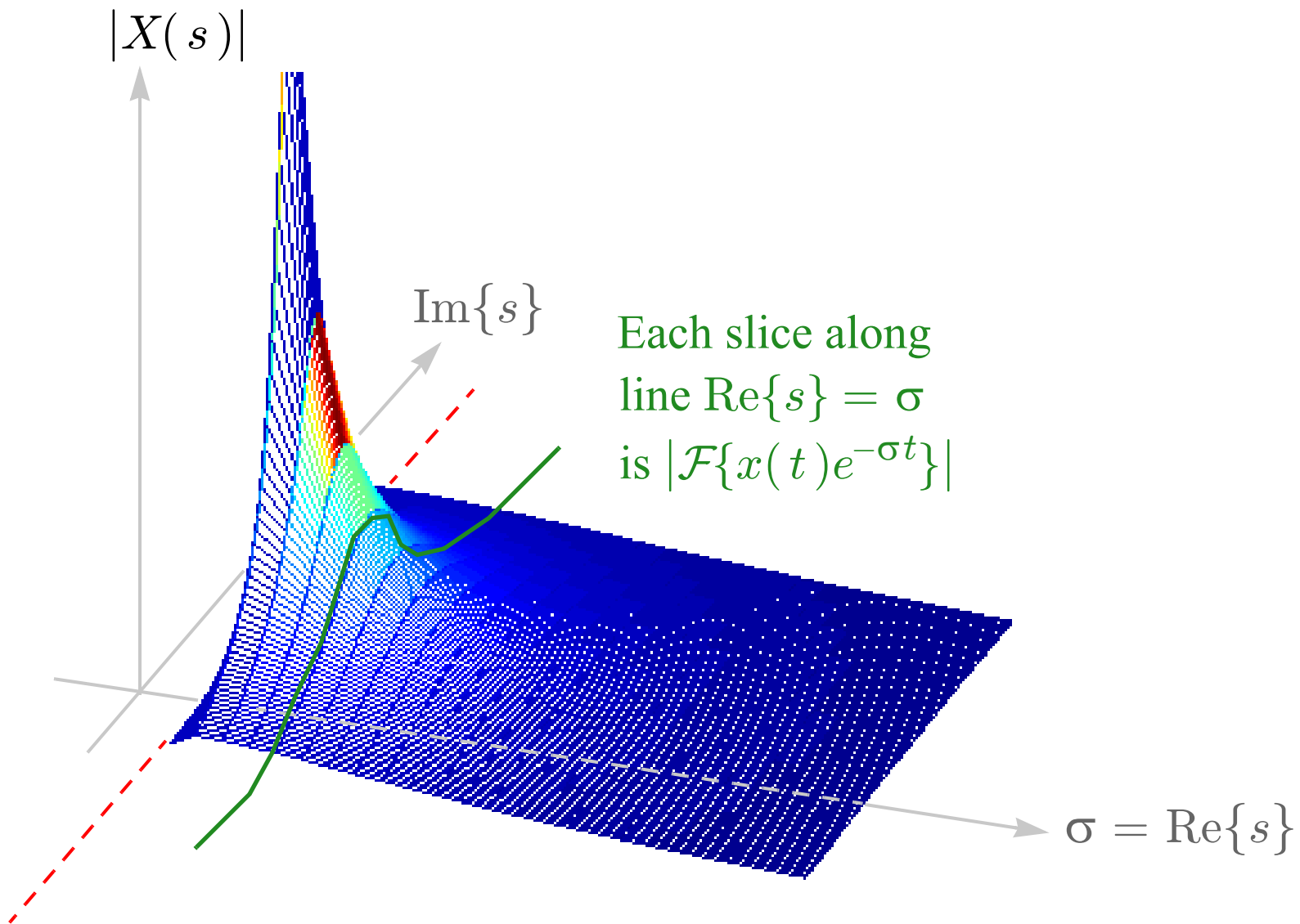
Useful interpretation relating Laplace to Fourier:

$$X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}, \text{ where } \sigma = \text{Re}\{s\}$$



FT integral will converge for large enough σ

Intuition



Pop Quiz

If $X(s) = \frac{1}{s^5}$, what is $x(t)$?

Hint: Consider this picture:

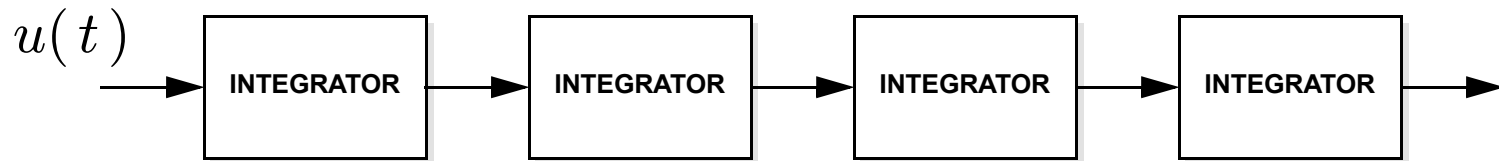


Table 1: Pairs

Impulse	$\delta(t)$	1	✓
Delayed Impulse	$\delta(t - t_0), \quad t_0 \geq 0$	e^{-st_0}	
Step	$u(t)$	$\frac{1}{s}$	✓
Rectangular Pulse	$u(t) - u(t - T), \quad T > 0$	$\frac{1 - e^{-sT}}{s}$	✓
Ramp	$tu(t)$	$\frac{1}{s^2}$	✓
Polynomial	$t^k u(t), \quad k \geq 0$	$\frac{k!}{s^{k+1}}$	✓
Exponential	$e^{-at}u(t)$	$\frac{1}{s + a}$	✓
Polynomial $\times$ Exponential	$t^k e^{-at}u(t)$	$\frac{k!}{(s + a)^{k+1}}$	
Cosine	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2 + \omega_0^2}$	
Sine	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	
Exponential $\times$ Cosine	$e^{-at} \cos(\omega_0 t)u(t)$	$\frac{s + a}{(s + a)^2 + \omega_0^2}$	
Exponential $\times$ Sine	$e^{-at} \sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s + a)^2 + \omega_0^2}$	

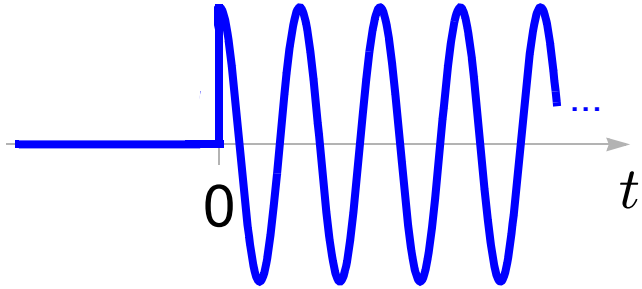
Table 2: Properties

Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$	✓
Right-Shift	$x(t - t_0), \quad t_0 \geq 0$	$e^{-st_0} X(s)$	✓
Time Scaling	$x(at), \quad a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$	
First Derivative	$\dot{x}(t) = \frac{d}{dt}x(t)$	$sX(s) - x(0)$	
Second Derivative	$\ddot{x}(t) = \frac{d^2}{dt^2}x(t)$	$s^2 X(s) - sx(0) - \dot{x}(0)$	
Integration	$\int_0^t x(\tau) d\tau$	$\frac{X(s)}{s}$	✓
Modulation	$x(t)e^{at}$	$X(s - a)$	
Convolution	$x(t) * h(t)$	$X(s)H(s)$	✓
Final Value Theorem	$\lim_{t \rightarrow \infty} x(t), \quad (\text{if limit exists})$	$\lim_{s \rightarrow 0} sX(s)$	
Initial Value Theorem	$\lim_{t \rightarrow 0} x(t), \quad (\text{if limit exists})$	$\lim_{s \rightarrow \infty} sX(s)$	

s -domain differentiation $-tx(t)$ $\frac{d}{ds}X(s)$

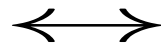
Motivate Linearity: One-Sided Cosine

$$x(t) = \cos(\omega_0 t) u(t) \quad \longleftrightarrow \quad X(s) =$$

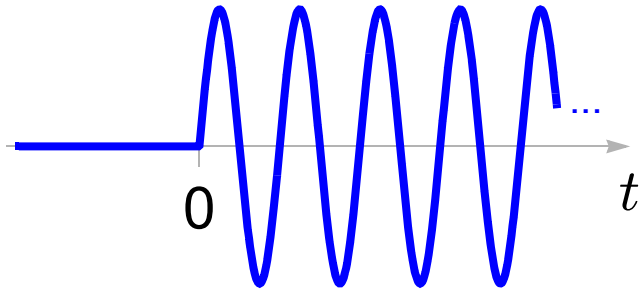


One-Sided Sine

$$x(t) = \sin(\omega_0 t) u(t)$$



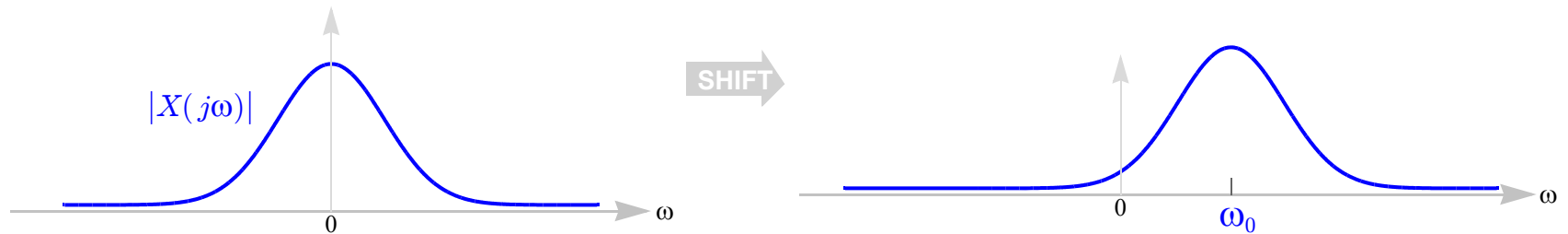
$$X(s) =$$



Modulation Property

In context of Fourier:

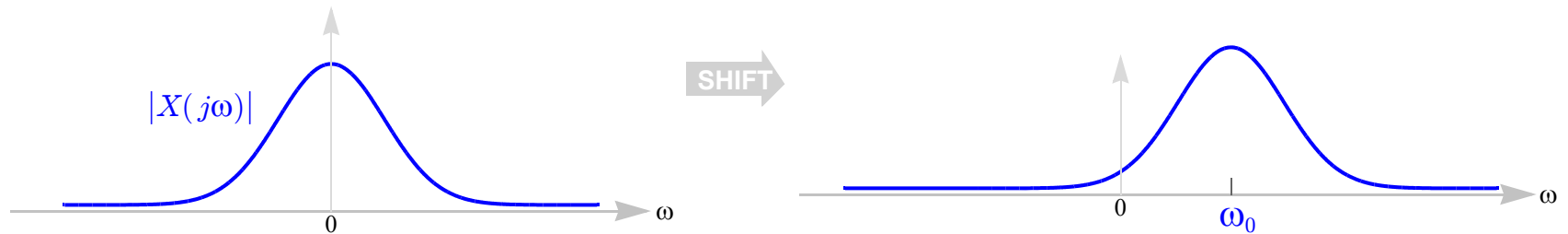
$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega - \omega_0))$$



Modulation Property

In context of Fourier:

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega - \omega_0))$$



In context of Laplace:

$$e^{at} x(t) \longleftrightarrow X(s - a)$$



Example using Tables

Prob. 1. Find LT of $x(t) = te^{-t}u(t)$ using 4 different techniques:

- ▷ integration by parts
- ▷ modulation property
- ▷ convolution property
- ▷ s -domain differentiation property

Poles and Zeros for Rational $X(s)$

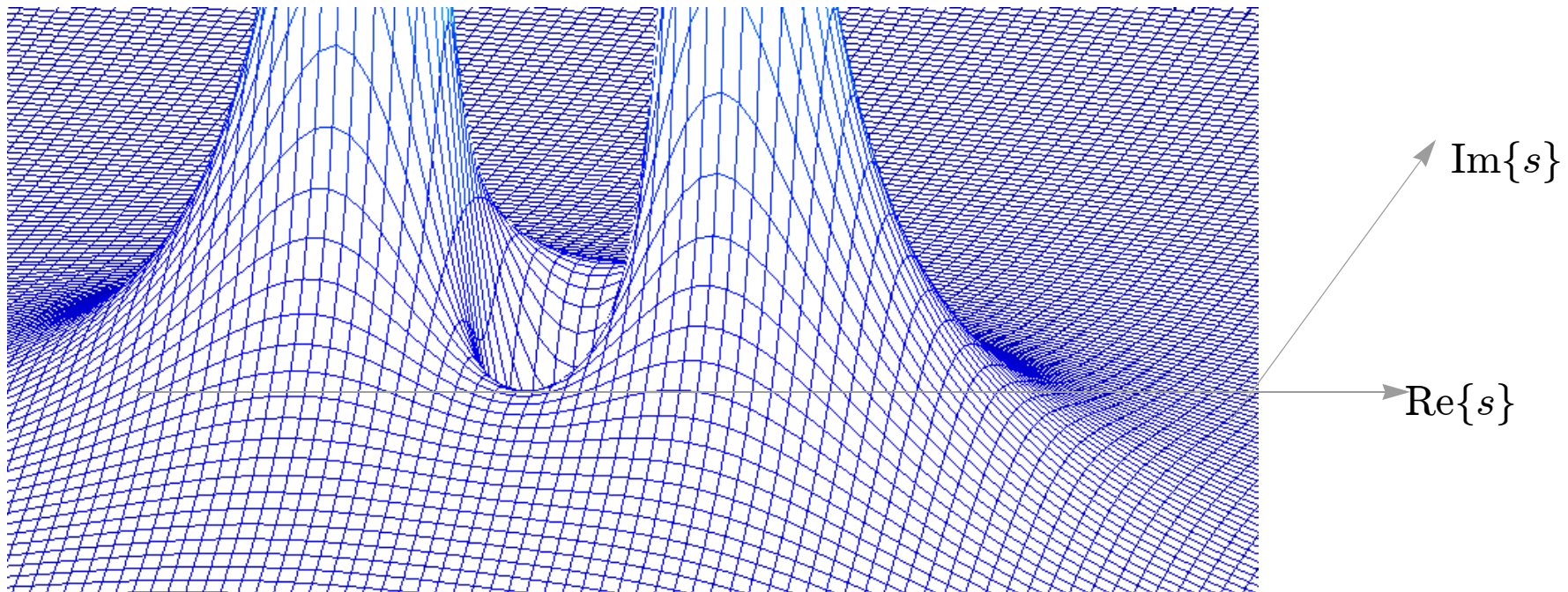
$$X(s) = \frac{P(s)}{Q(s)}$$

up to a scaling constant, fully specified by:

“zeros” = roots of numerator polynomial

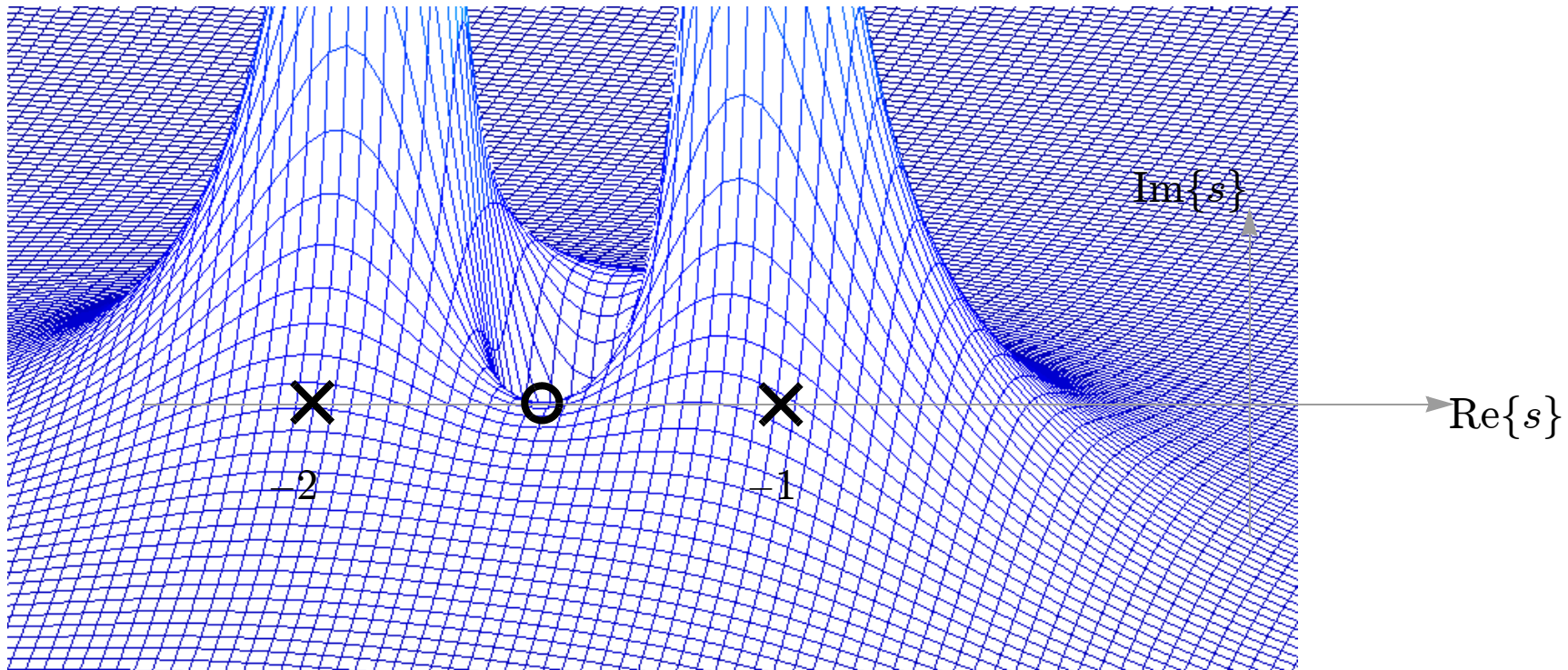
“poles” = roots of denominator polynomial

Example: The magnitude surface for $X(s) = \frac{s + 1.5}{s^2 + 3s + 2}$ looks like this:



Example

Comparing pole-zero plot for $X(s) = \frac{s + 1.5}{s^2 + 3s + 2}$ to the $|X(s)|$ surface:



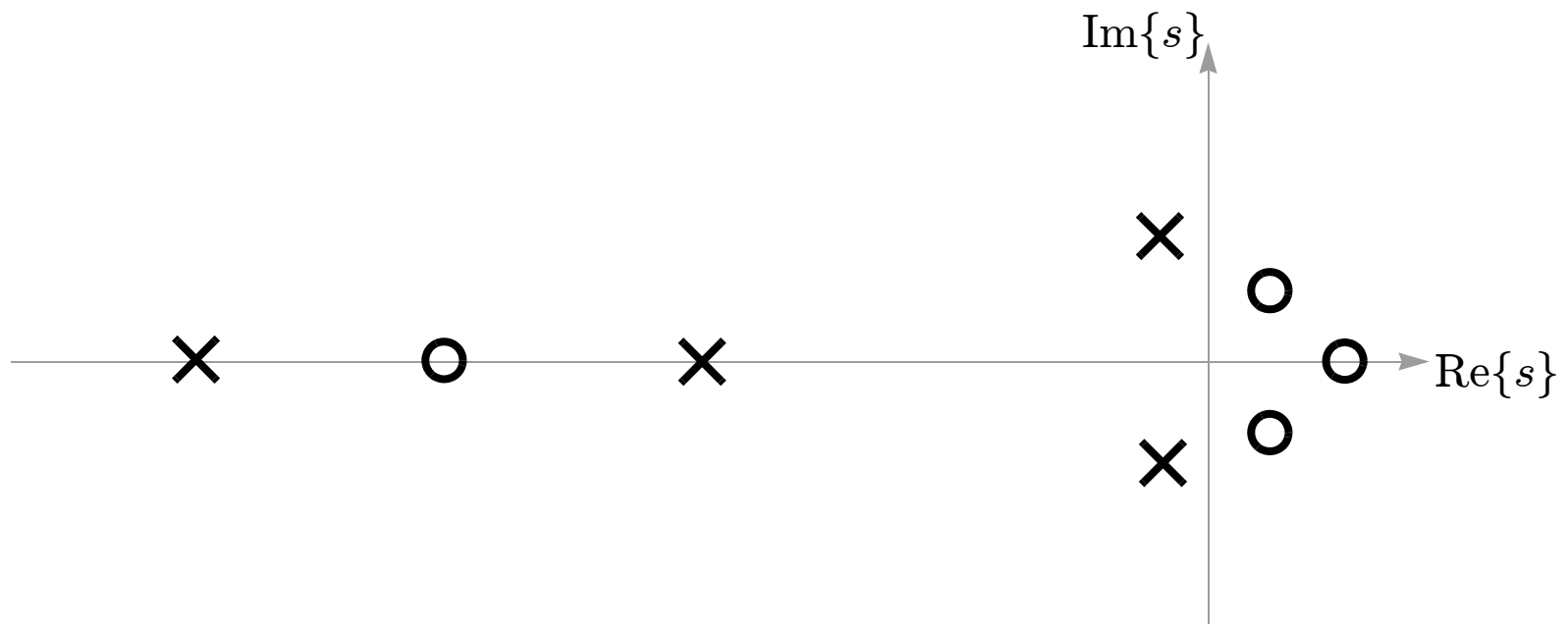
ROC

ROC = Values of s for which integral $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$ converges

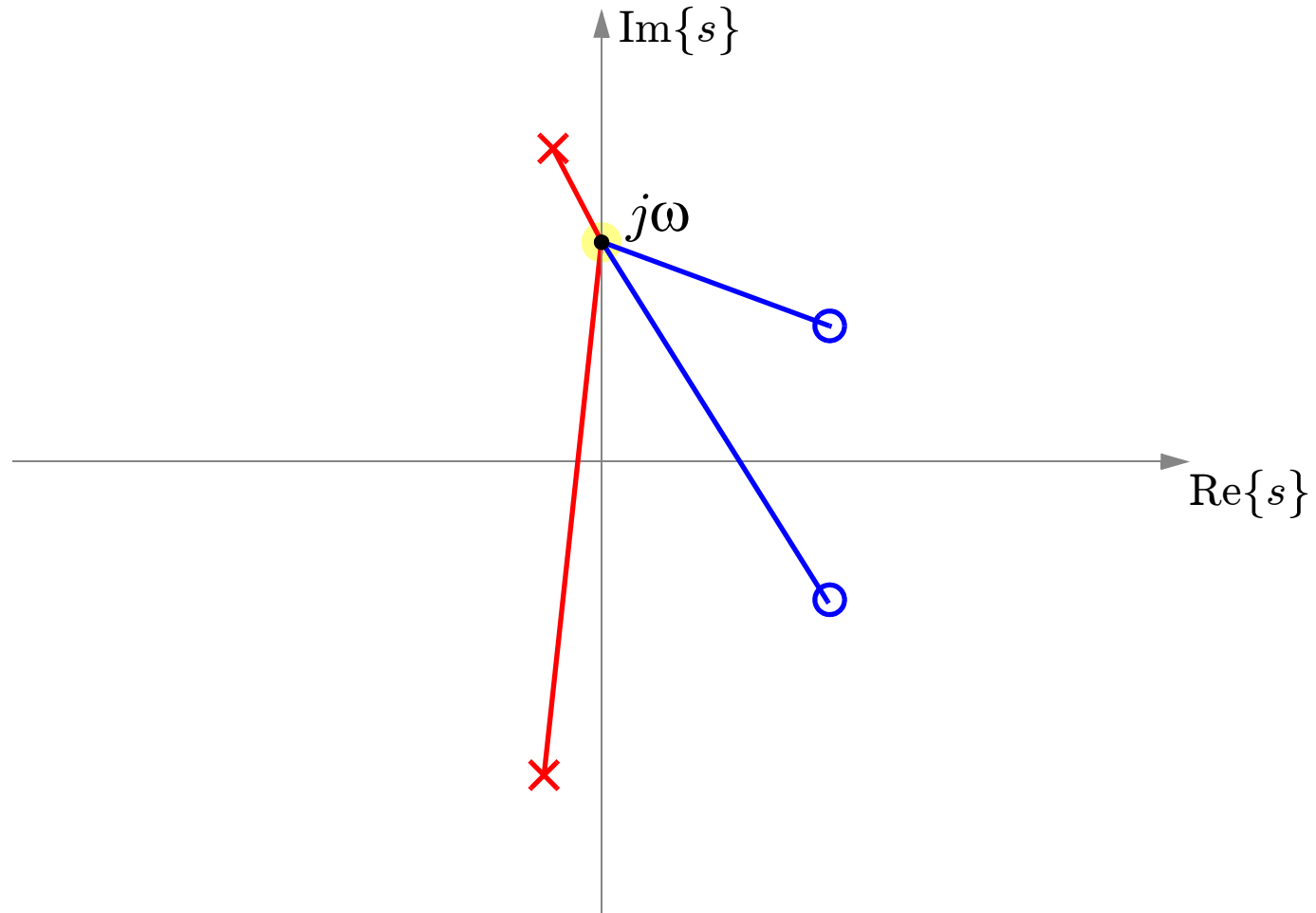
Properties:

- $\sigma_0 \in \text{ROC} \Rightarrow \sigma > \sigma_0 \in \text{ROC}$
- rational $\Rightarrow$ ROC can't include poles
- Combine last two $\Rightarrow$ for causal signals: ROC $>$ rightmost pole
- Fourier transform exists $\Rightarrow$ ROC includes imaginary axis
- Combine last two: all poles of causal signal in LHP $\Rightarrow$ FT exist

Causal & Stable $\Rightarrow$ All Poles in LHP



Geometry of FT and Pole-Zero Plot



What Kind of Filter is This?

$$H(s) = \frac{s^2 + 1}{s^2 + 0.2s + 1.01}$$