Lecture 16: Thu Oct 8, 2020

Reminder:

- Homework 6 due tonight at midnight

Lecture

- finish MF pop quiz
- Gaussian pulses:
    - ▹ same shape in time domain and frequency domain
    - ▹ maximally concentrated in time and frequency
- duration-bandwidth product
- uncertainty principle
- AM modulation

TABLE 1

Table of Fourier Transform Pairs		
	Time-Domain: $x(t)$	*Frequency-Domain:* $X(j\omega)$
Right-sided exponential	$e^{-at}u(t) \quad (a > 0)$	$\dfrac{1}{a + j\omega}$
Left-sided exponential	$e^{bt}u(-t) \quad (b > 0)$	$\dfrac{1}{b - j\omega}$
Square pulse	$[u(t + T/2) - u(t - T/2)]$	$\dfrac{\sin(\omega T/2)}{\omega/2}$
"sinc" function	$\dfrac{\sin(\omega_0 t)}{\pi t}$	$[u(\omega + \omega_0) - u(\omega - \omega_0)]$
Impulse	$\delta(t)$	1
Shifted impulse	$\delta(t - t_0)$	$e^{-j\omega t_0}$
Complex exponential	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
General cosine	$A\cos(\omega_0 t + \phi)$	$\pi A e^{j\phi}\delta(\omega - \omega_0) + \pi A e^{-j\phi}\delta(\omega + \omega_0)$
Cosine	$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin(\omega_0 t)$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$
General periodic signal	$\displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$
Impulse train	$\displaystyle\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\dfrac{2\pi}{T} \displaystyle\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/T)$

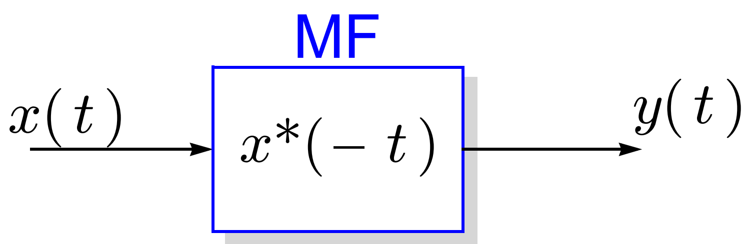
TABLE 2

Table of Fourier Transform Properties

	Time-Domain $x(t)$	*Frequency-Domain* $X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$ ✓
Conjugation	$x^*(t)$	$X^*(-j\omega)$ ✓
Time-Reversal	$x(-t)$	$X(-j\omega)$ ✓
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}X(j(\omega/a))$ ✓
Delay	$x(t-t_d)$	$e^{-j\omega t_d}X(j\omega)$ ✓
Modulation	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$ ✓
Modulation	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$ ✓
Differentiation	$\frac{d^k x(t)}{dt^k}$	$(j\omega)^k X(j\omega)$
Convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$ ✓
Multiplication	$x(t)p(t)$	$\frac{1}{2\pi}X(j\omega) * P(j\omega)$ ✓

Pop Quiz

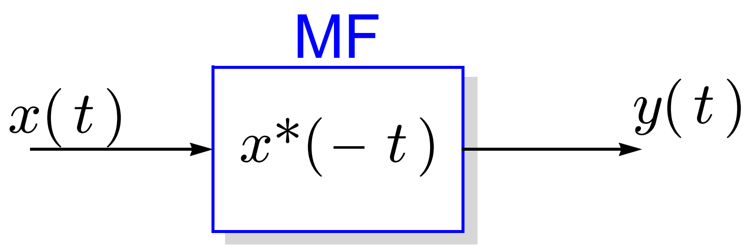
(a) Express output FT $Y(j\omega)$ in terms of input FT $Y(j\omega)$:



(b) Evaluate output at time zero: What is $y(0)$?

Pop Quiz

(a) Express output FT $Y(j\omega)$ in terms of input FT $Y(j\omega)$:

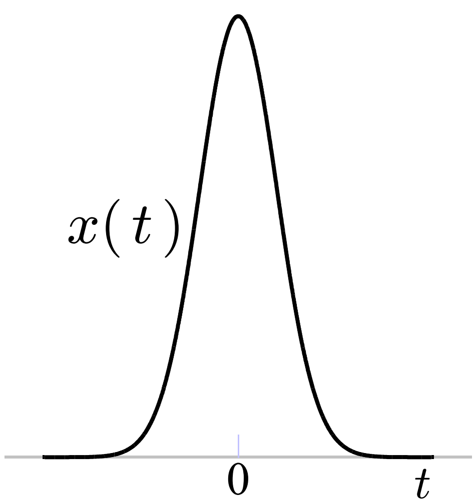


(b) Evaluate output at time zero: What is $y(0)$?

$\Rightarrow$ **Parseval:** $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Gaussian FT Pair

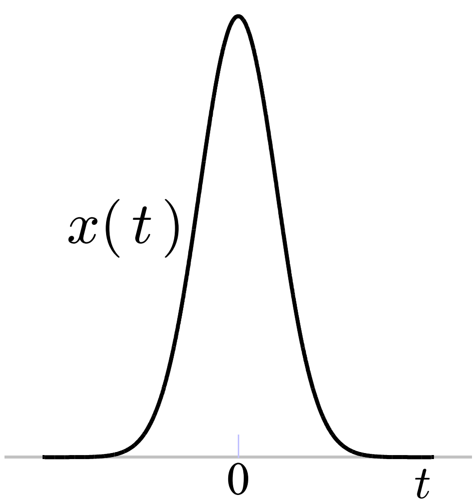
$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)} \quad \longleftrightarrow \quad ?$$

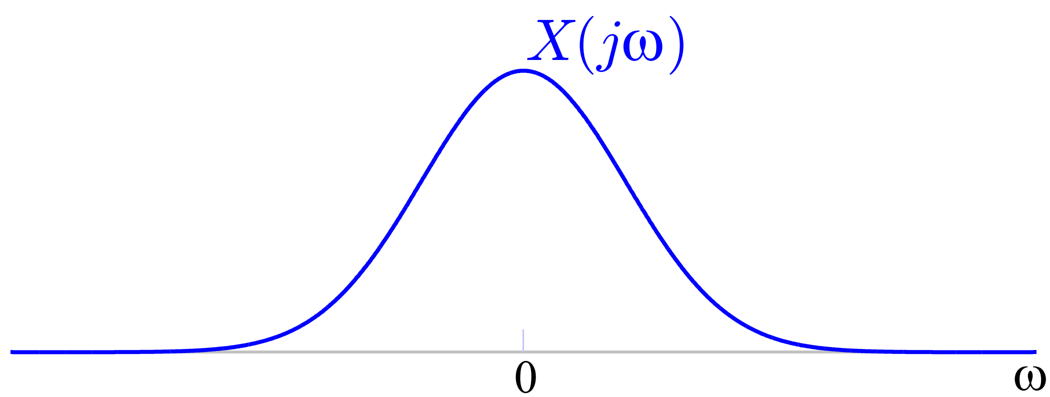


Gaussian FT Pair

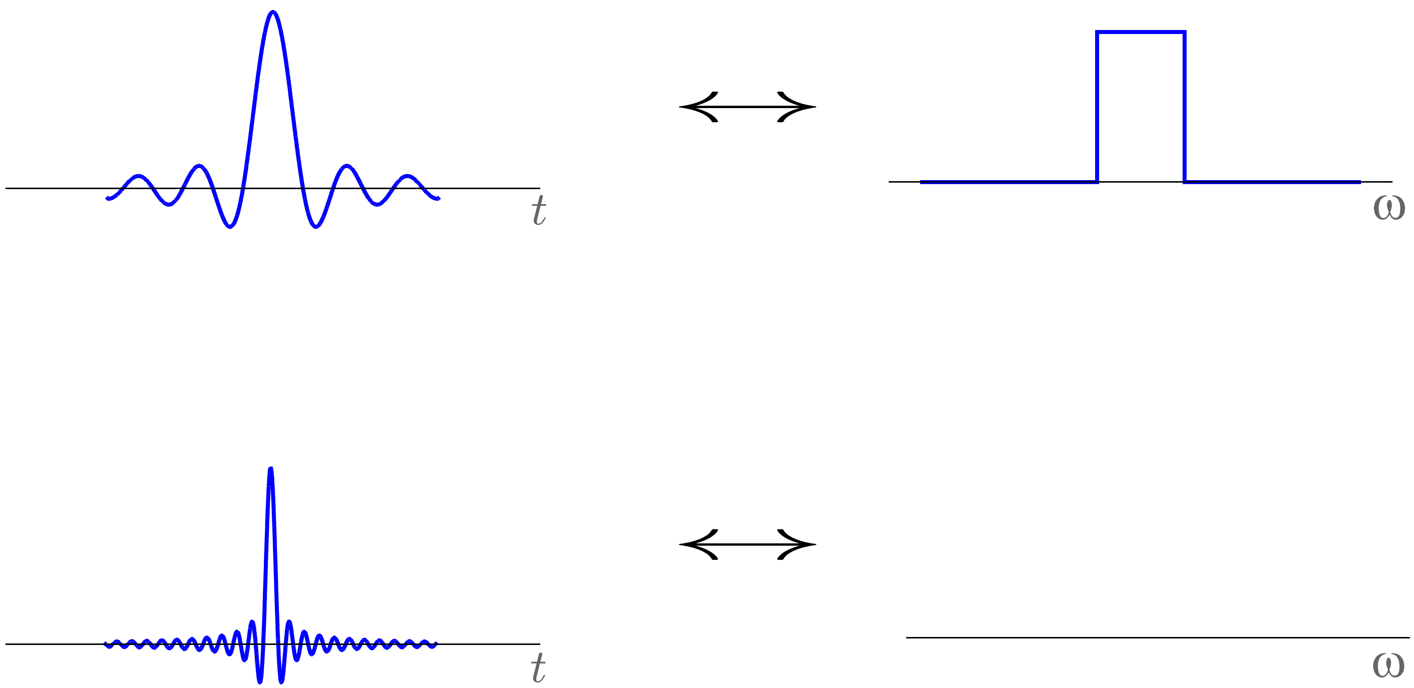
Same shape in both time and frequency,
and *maximally concentrated* in both time and frequency:

$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)} \quad \longleftrightarrow \quad X(j\omega) = e^{-\sigma^2\omega^2/2}$$

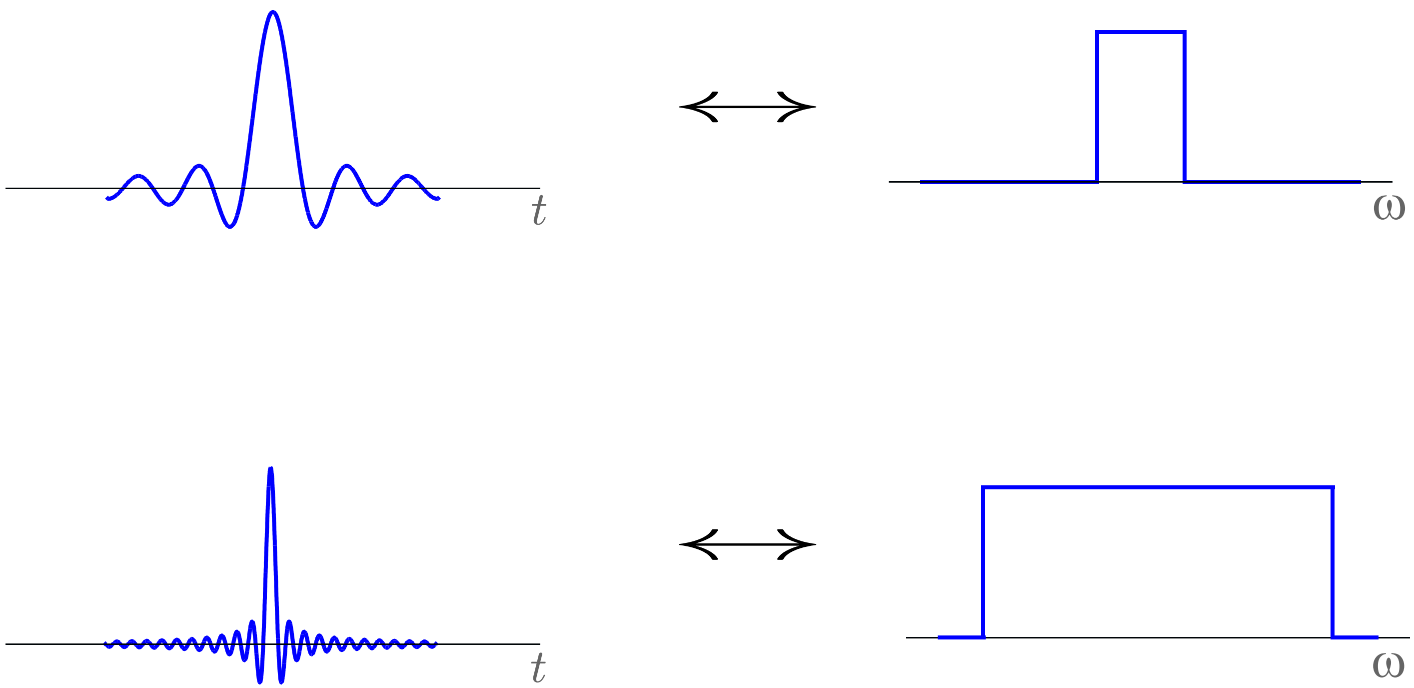




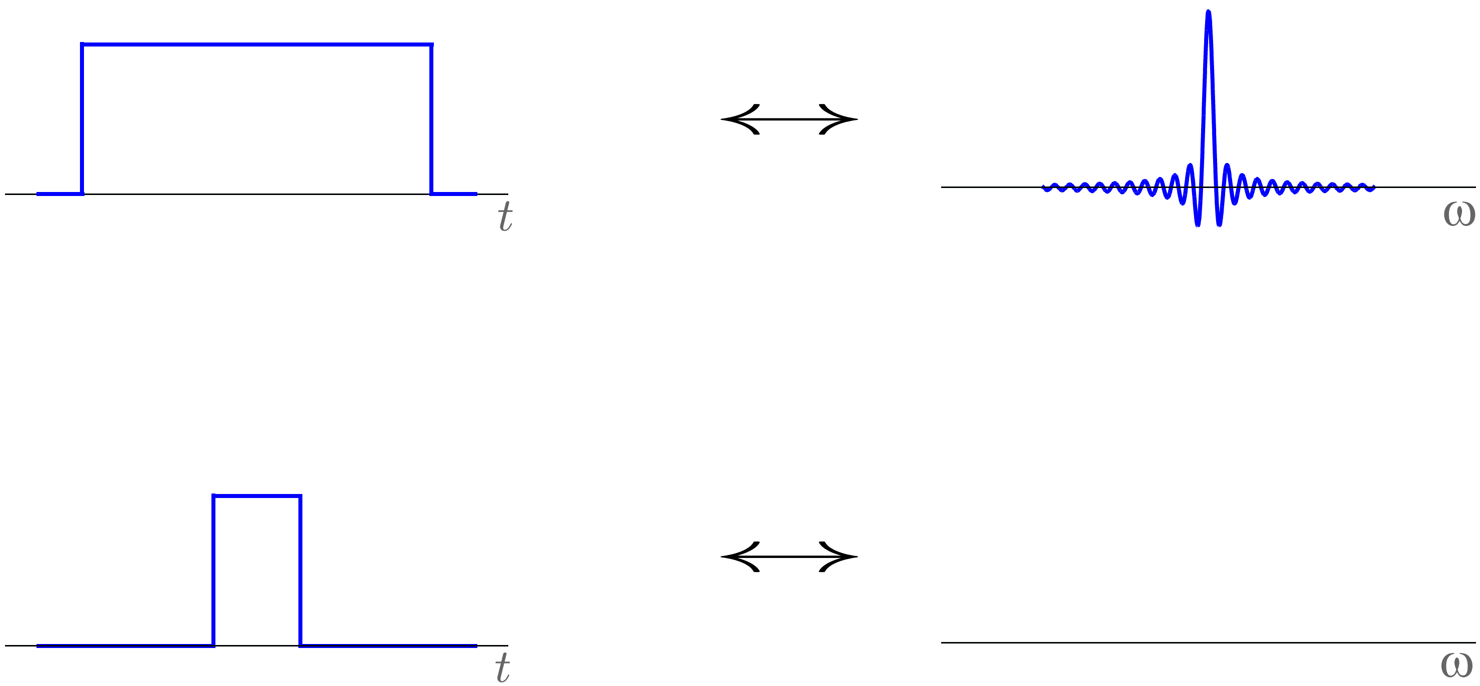
Long vs Short: Sinc Pulse



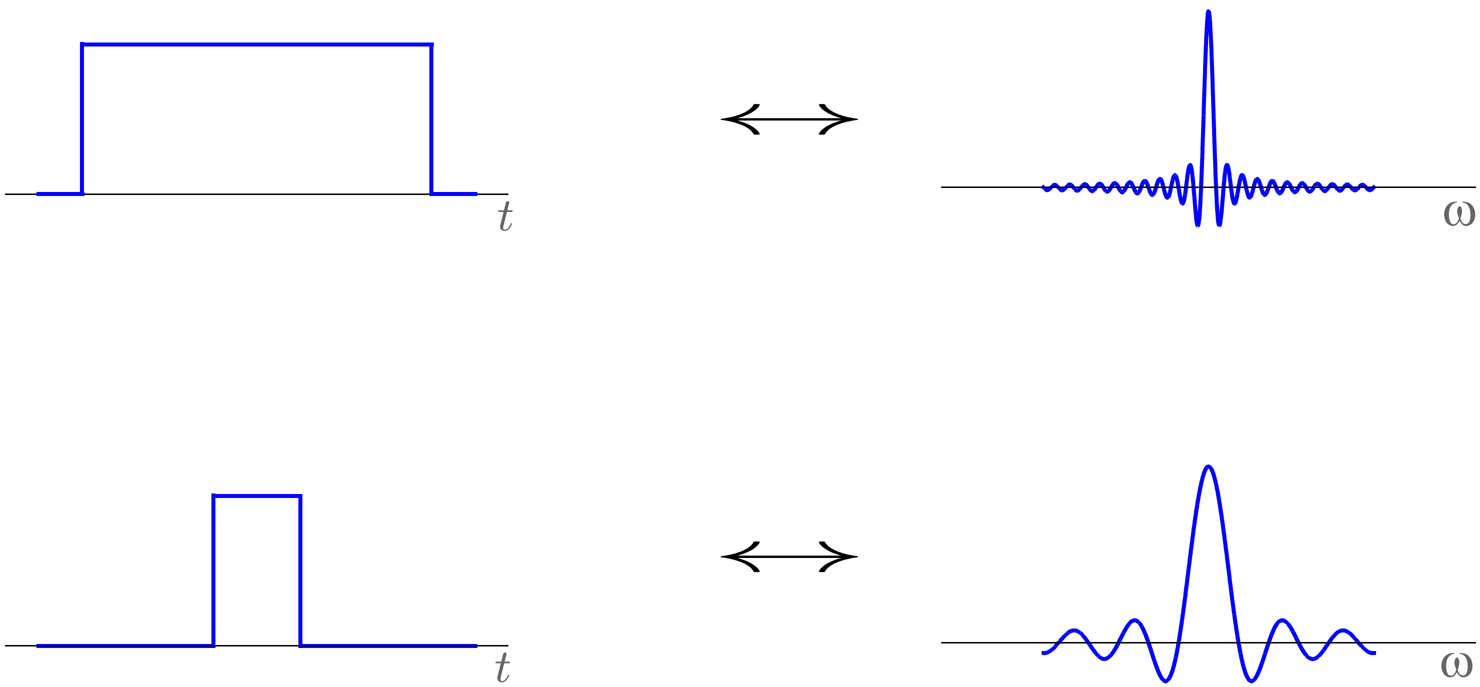
Long vs Short: Sinc Pulse



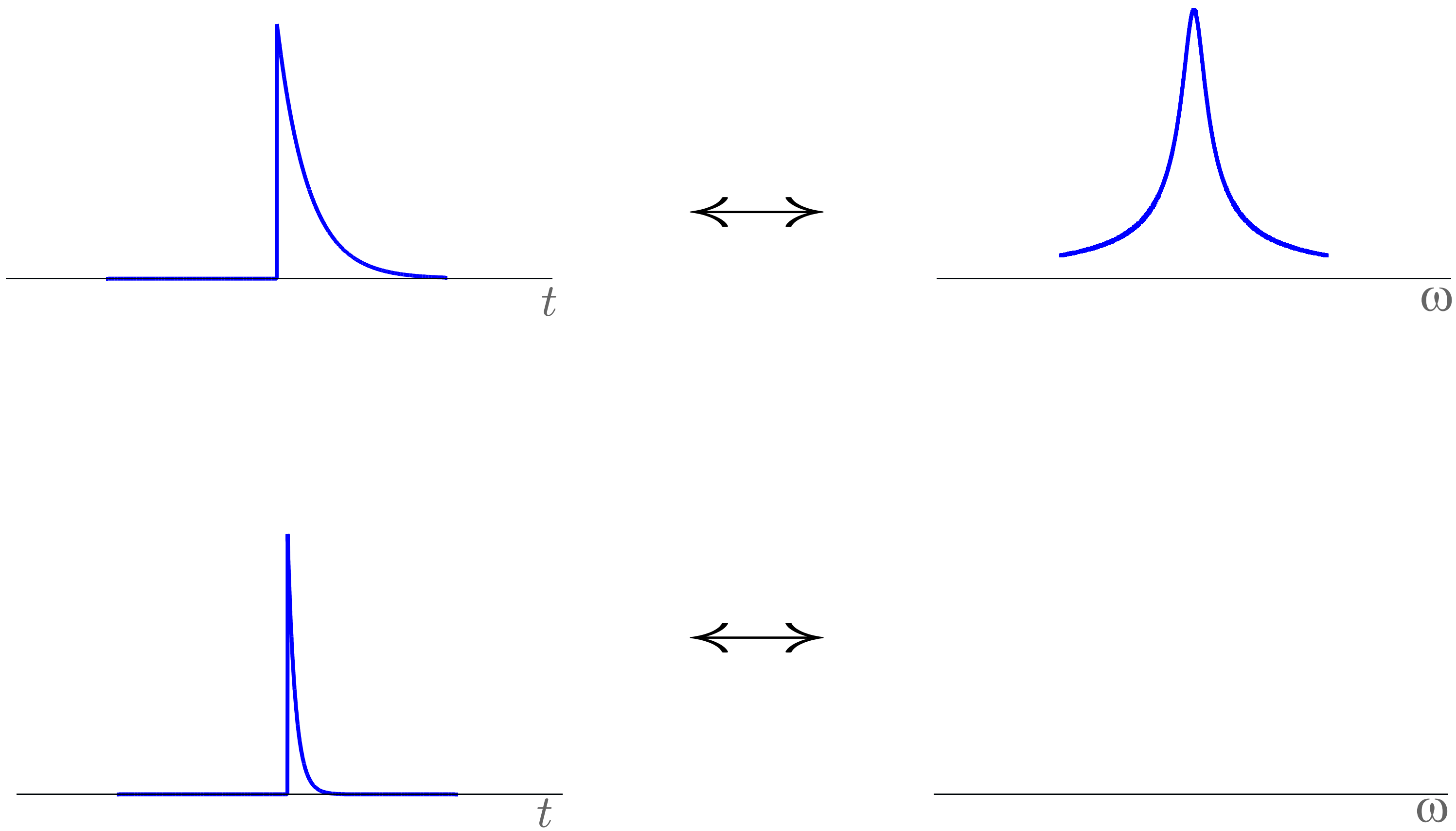
Long vs Short: Rectangular Pulse



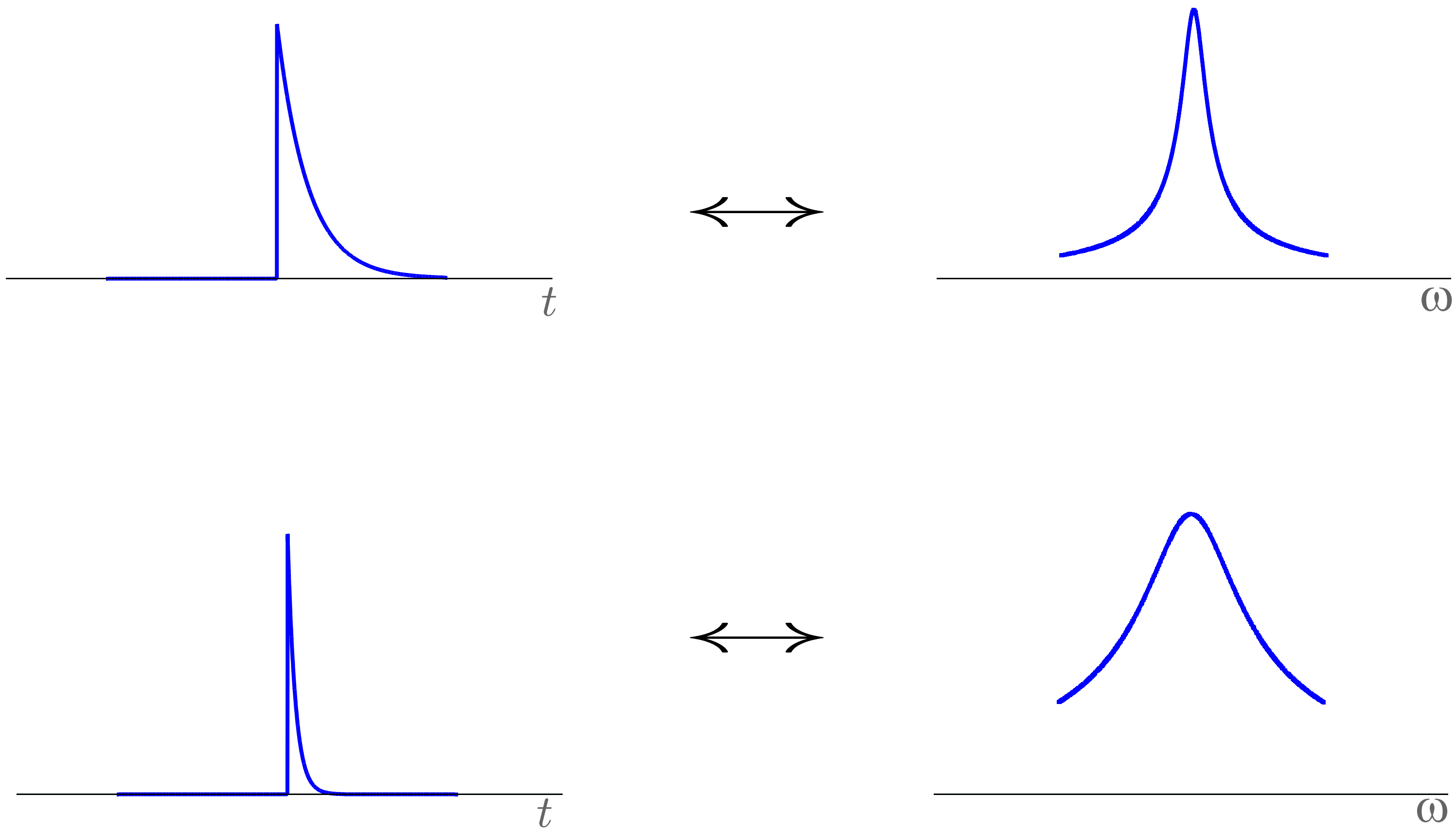
Long vs Short: Rectangular Pulse



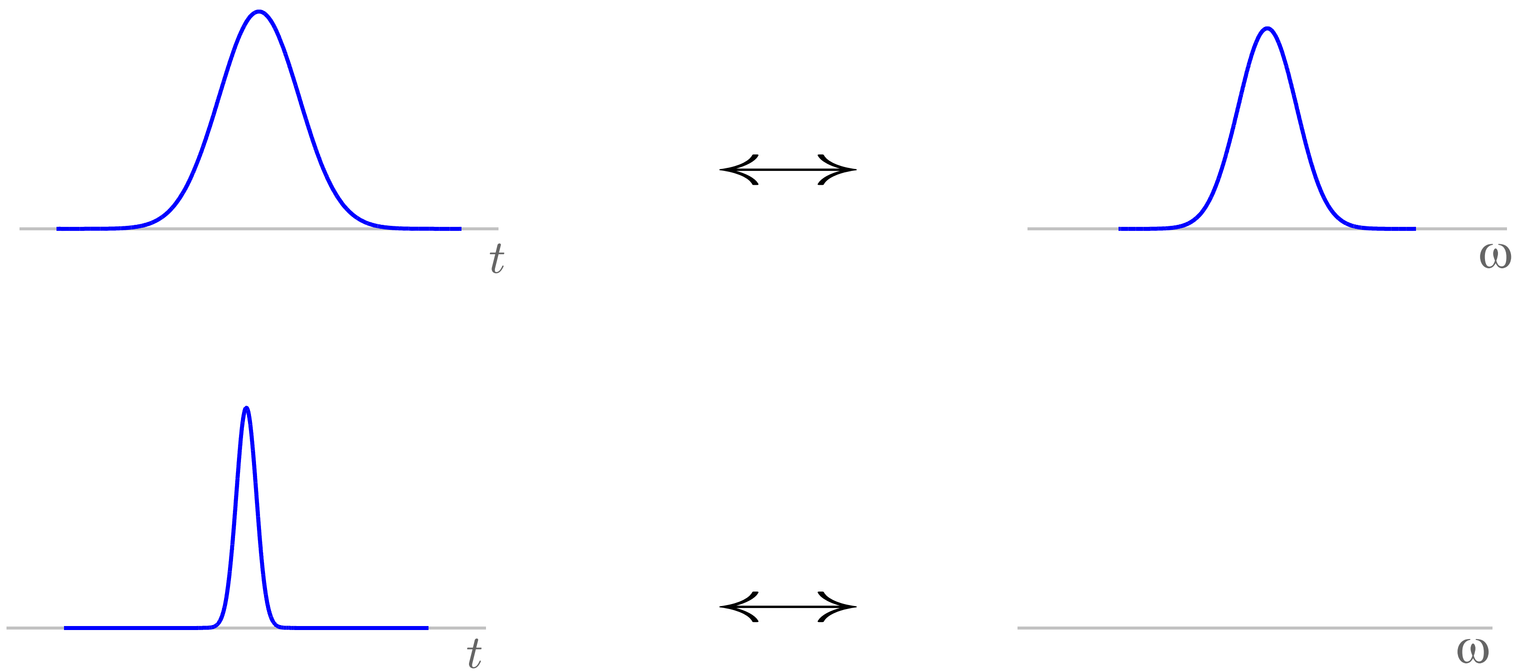
Long vs Short: Exponential Pulse



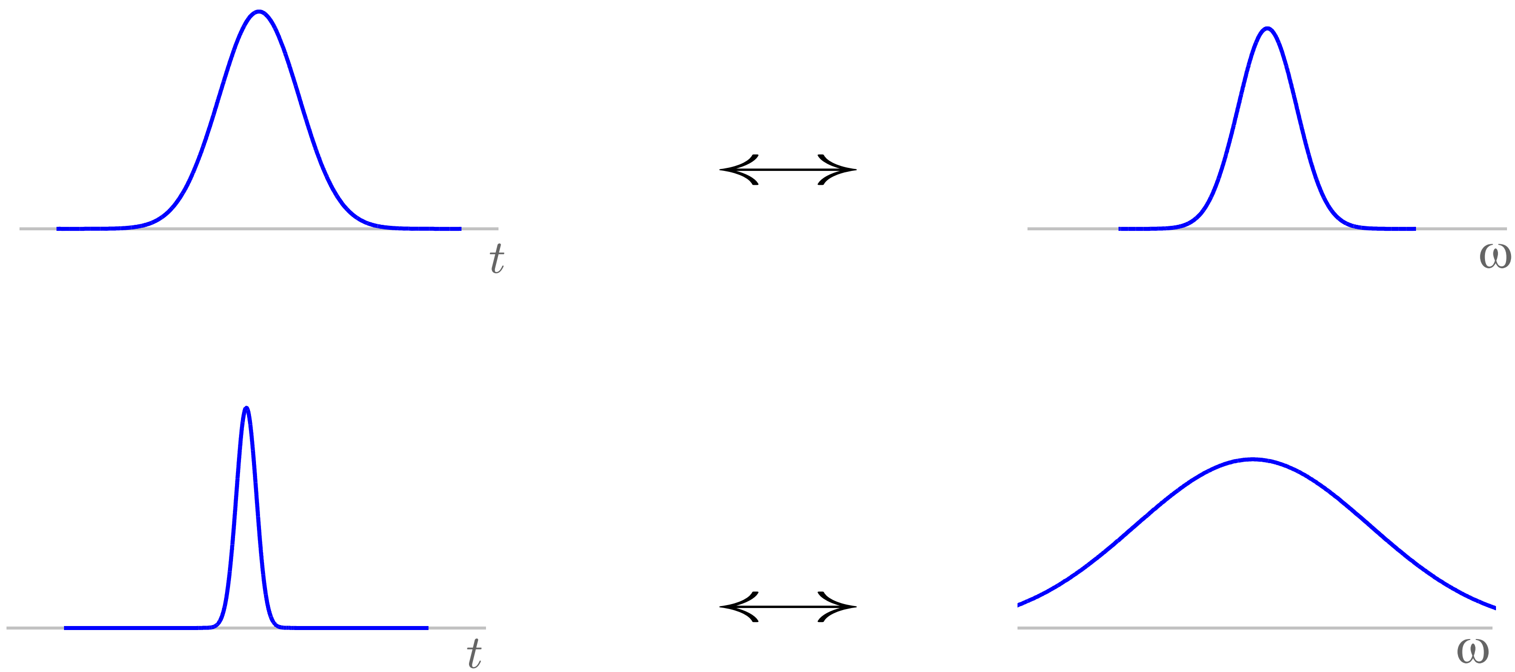
Long vs Short: Exponential Pulse



Long vs Short: Gaussian Pulse



Long vs Short: Gaussian Pulse



Uncertainty Principle

In physics (Heisenberg),
where position & momentum probability distributions form a FT pair:

Particle position and momentum cannot both be known precisely

$\Rightarrow$ It follows from a fundamental result of Fourier transforms:

A signal cannot be simultaneously narrow in time and in frequency:

$$\text{(DURATION)(BANDWIDTH)} \geq \text{constant}$$

The value of the constant depends on how “duration” and “bandwidth” are defined.

Pop Quiz: True or False

(a) Short duration implies large bandwidth

(b) Long duration implies small bandwidth

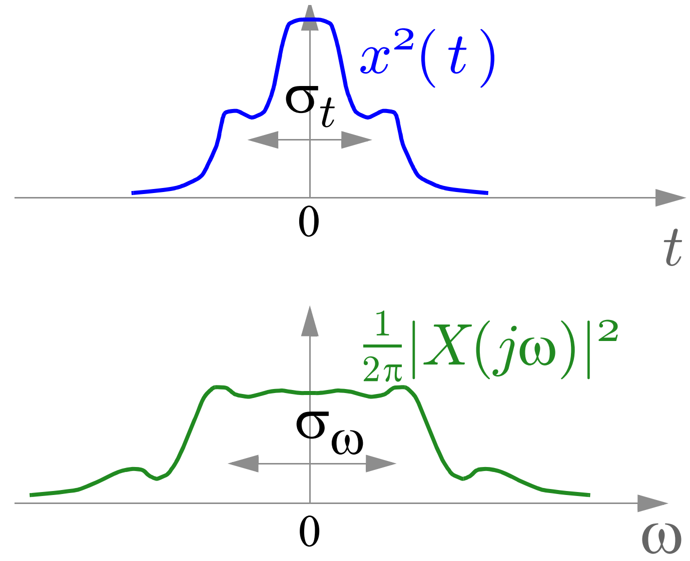
Any Loss of Generality?

Assume

- ▷ $x(t)$ is unit energy
- ▷ $x(t)$ is "centered" at time 0, *i.e.*, delayed so that $\int_{-\infty}^{\infty} t x^2(t) dt = 0$
- ▷ $X(j\omega)$ is "centered" at 0, *i.e.*, shifted so that $\int_{-\infty}^{\infty} \omega |X(j\omega)|^2 \, d\omega = 0$

(free when $x(t)$ is real.)

The pdf View



- Interpret $x^2(t)$ and $\frac{1}{2\pi}|X(j\omega)|^2$ as "pdf's"; they both are:
  - ▷ positive
  - ▷ integrate to 1

- Define duration σ_t and bandwidth σ_ω using corresponding *variance*:

$$\sigma_t^2 = \int_{-\infty}^{\infty} t^2 x^2(t)\,dt,$$

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega.$$

With these definitions, the uncertainty principle is: $\sigma_t \sigma_\omega \geq \frac{1}{2}$.

Equality iff $x(t) = Ae^{-Bt^2}$ = "Gaussian".

Proof of Uncertainty Principle

Integrating by parts: $$1 = \int_{-\infty}^{\infty} \underset{u}{x^2(t)}\,\underset{dv}{dt} = uv - \int v\,du$$

$$= \cancel{tx^2(t)\Big|_{-\infty}^{\infty}} - 2\int_{-\infty}^{\infty} t x(t)\dot{x}(t)\,dt$$

Square both sides

$$\Rightarrow \frac{1}{4} = \left(\int_{-\infty}^{\infty} t x(t)\dot{x}(t)\,dt\right)^2 \underset{\text{CAUCHY SCHWARZ}}{\leq} \int_{-\infty}^{\infty} t^2 x^2(t)\,dt \int_{-\infty}^{\infty} \dot{x}(t)^2\,dt$$

$$= \sigma_t^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|j\omega X(j\omega)\right|^2 d\omega$$

$$= \sigma_t^2 \sigma_\omega^2 \qquad \Rightarrow \qquad \boxed{\sigma_t \sigma_\omega \geq 0.5.}$$

From C-S, equality when $\dot{x}(t) \propto t x(t) \quad \Rightarrow \quad x(t) = Ae^{-Bt^2}$ "Gaussian"

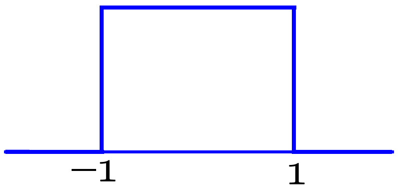
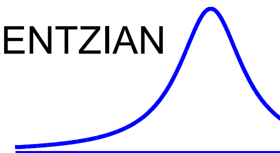
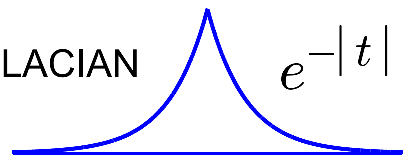
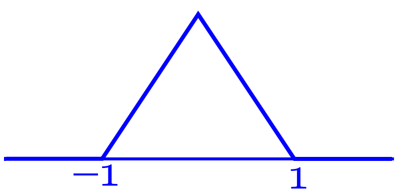
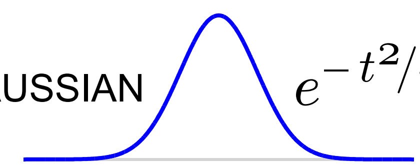
Cauchy Schwarz Intuition

Correlation between two *different* signals with identical energy cannot exceed energy of just one:

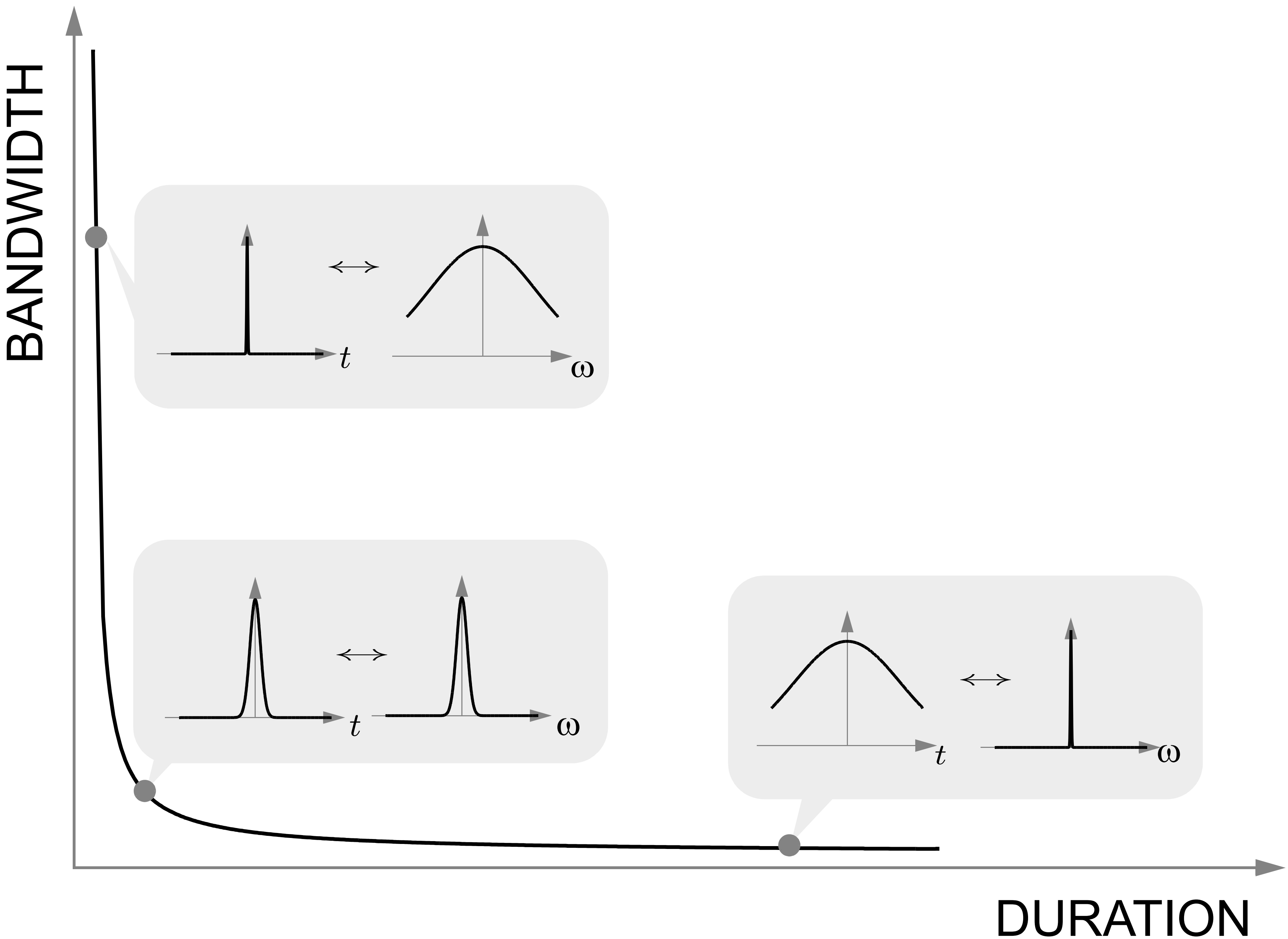
$$\int_{-\infty}^{\infty} \frac{x(t)}{\sqrt{E_x}} \frac{y(t)}{\sqrt{E_y}} dt \quad \leq \quad \int_{-\infty}^{\infty} \frac{x(t)}{\sqrt{E_x}} \frac{x(t)}{\sqrt{E_x}} dt \quad = \quad 1.$$

Equality when?

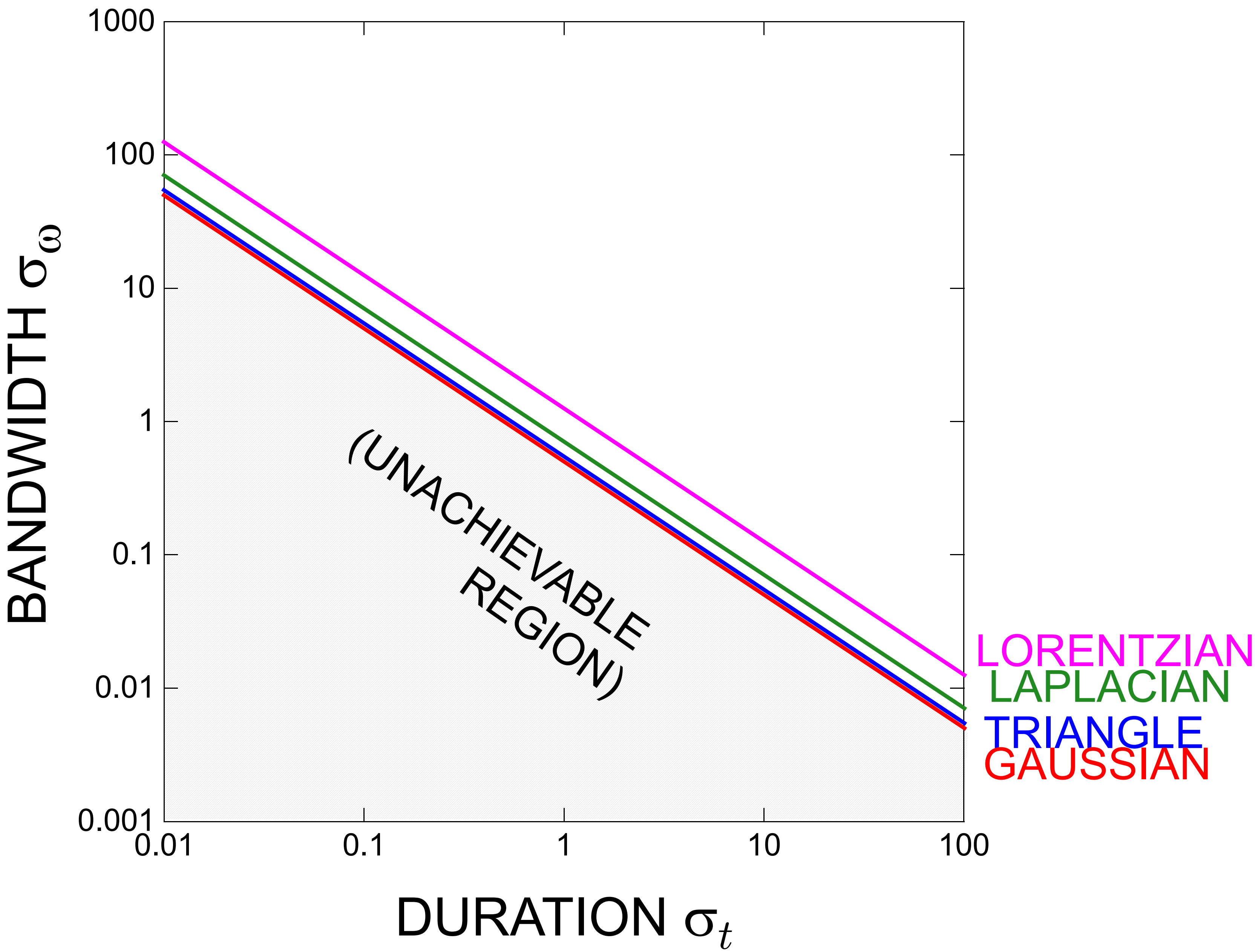
Examples

$x(t)$	DURATION σ_t	BANDWIDTH σ_ω	$\sigma_t\sigma_\omega$
	$1/\sqrt{3}$	∞	∞
LORENTZIAN  $\frac{1}{1+t^2}$	1	$\sqrt{\frac{\pi}{2}}$	1.253
LAPLACIAN  $e^{-\lvert t \rvert}$	$1/\sqrt{2}$	1	0.707
	$1/\sqrt{10}$	$\sqrt{3}$	0.548
GAUSSIAN  $e^{-t^2/4}$	1	$\frac{1}{2}$	$\frac{1}{2}$

BANDWIDTH vs DURATION



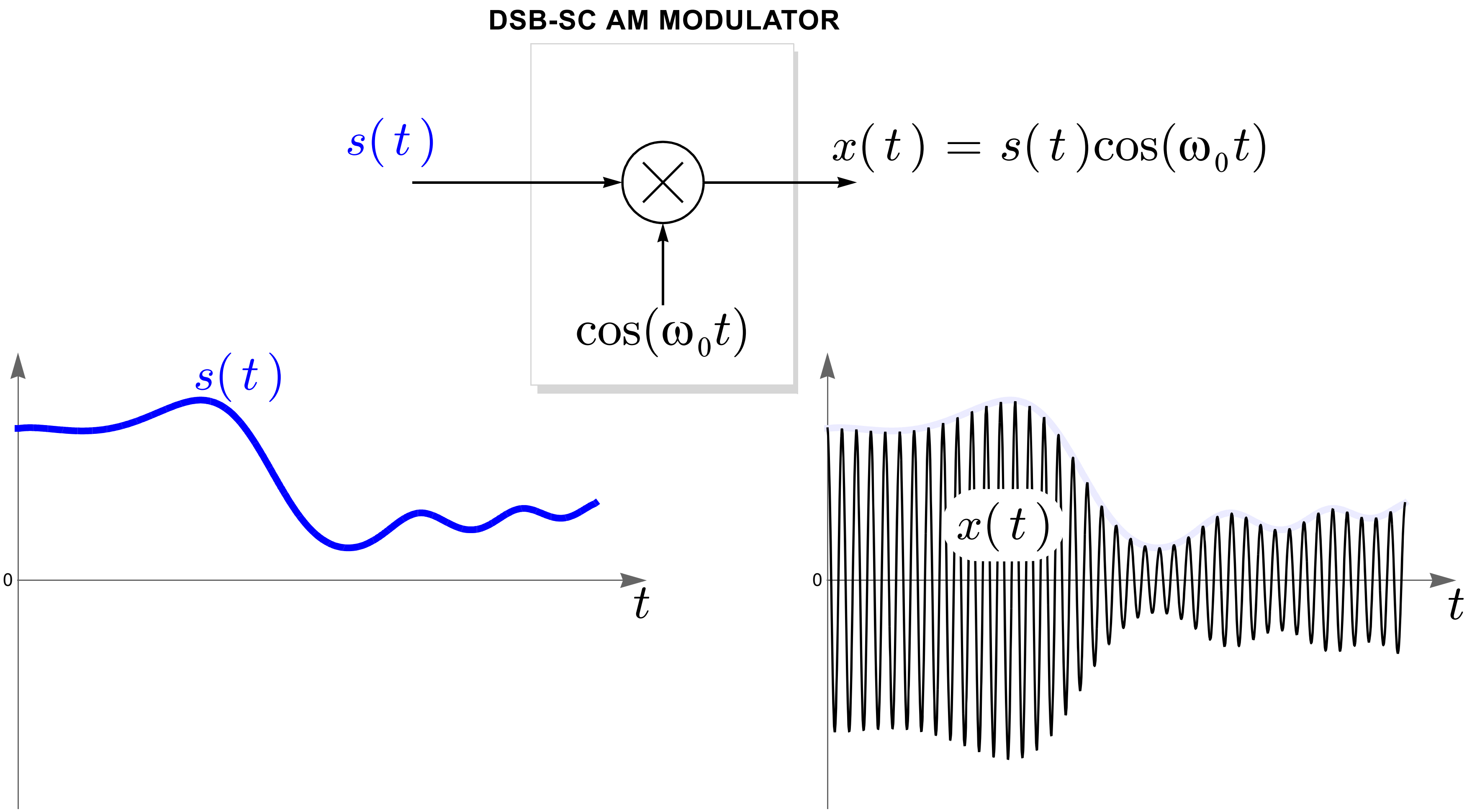
BANDWIDTH vs DURATION



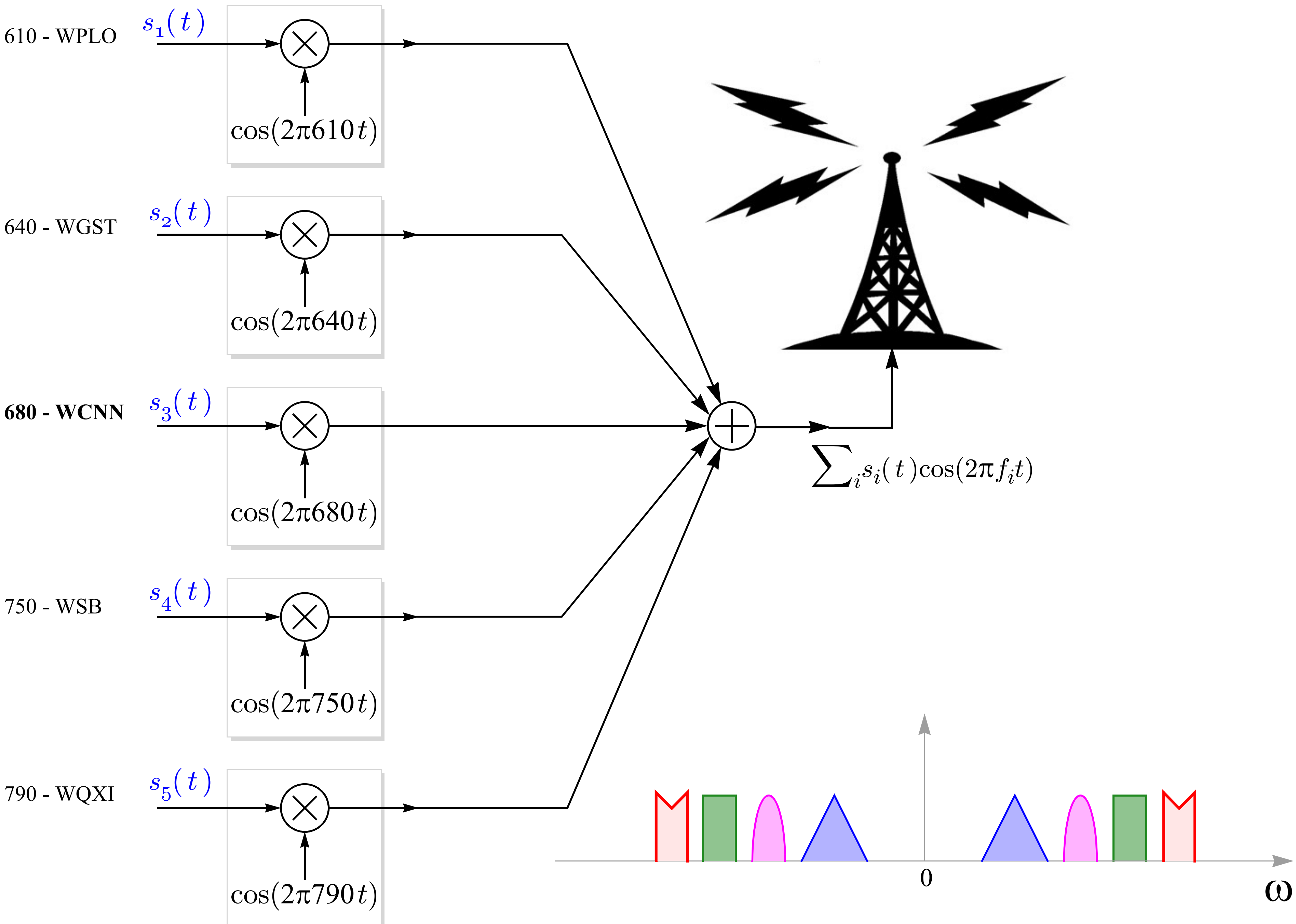
AM Radio

550 - WDUN - Talk
590 - WDWD - Radio Disney
610 - WPLO - Foreign Language Music
640 - WGST - Neocon Talk
680 - WCNN - Sports
750 - WSB - Neocon Talk
790 - WQXI - Sports Talk
860 - WAEC - Religion
920 - WGKA - Religious Talk
960 - WRFC - Talk
1010 - WGUN - Religion
1050 - WPBS - Religion
1060 - WKNG - Country
1080 - WFTD - Foreign Language Music
1100 - WWWE - Foreign Language Music
1160 - WCFO - Business Radio
1170 - WMLB - Adult Standards
1190 - WAFS - Foreign Language
1230 - WFOM - Neocon Talk
1260 - WTJH - Religion
1290 - WCHK - Talk
1340 - WALR - Foreign Language
1380 - WAOK - Talk
1400 - WLTA - Religion
1410 - WKKP - Adult Standards
1420 - WATB - Multicultural
1460 - WXEM - Foreign Language
1480 - WYZE - Religion
1500 - WDPC - Religion
1520 - WDCY - Religion
1550 - WAZX - Foreign Language
1570 - WSSA - Religion
1580 - WKUN - Country
1600 - WAOS - Foreign Language
1690 - WMLB - Voice of the Arts

AM Modulation

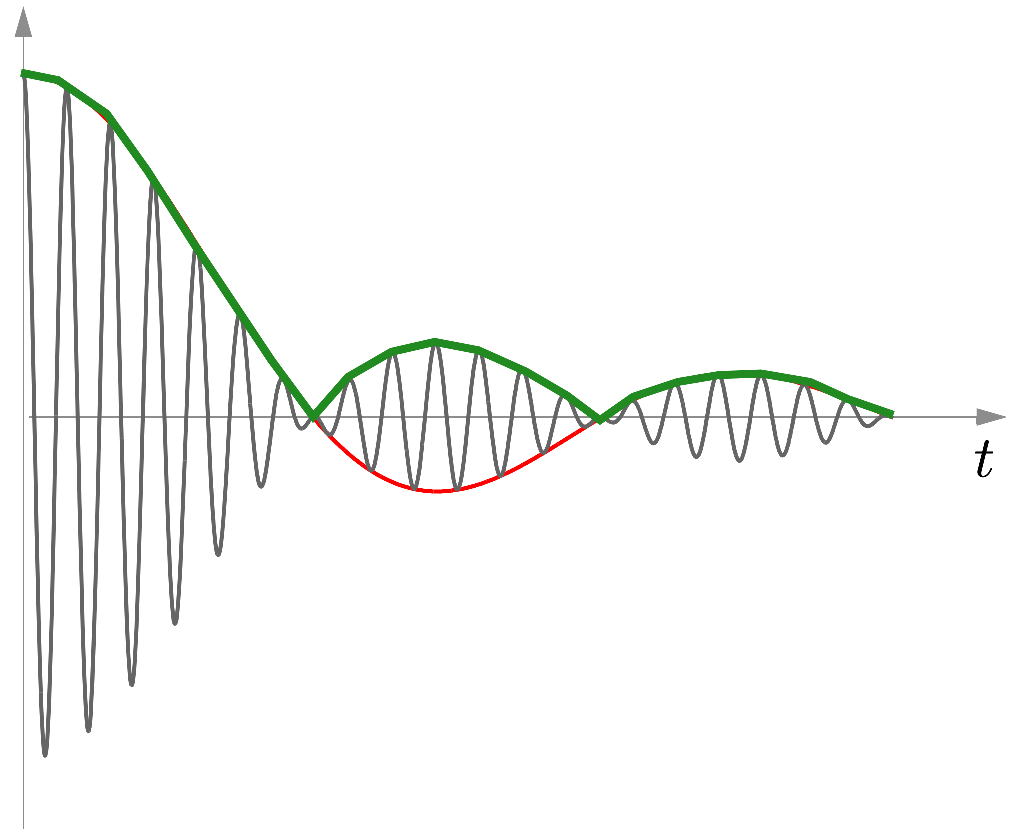


Frequency-Division Multiplexing



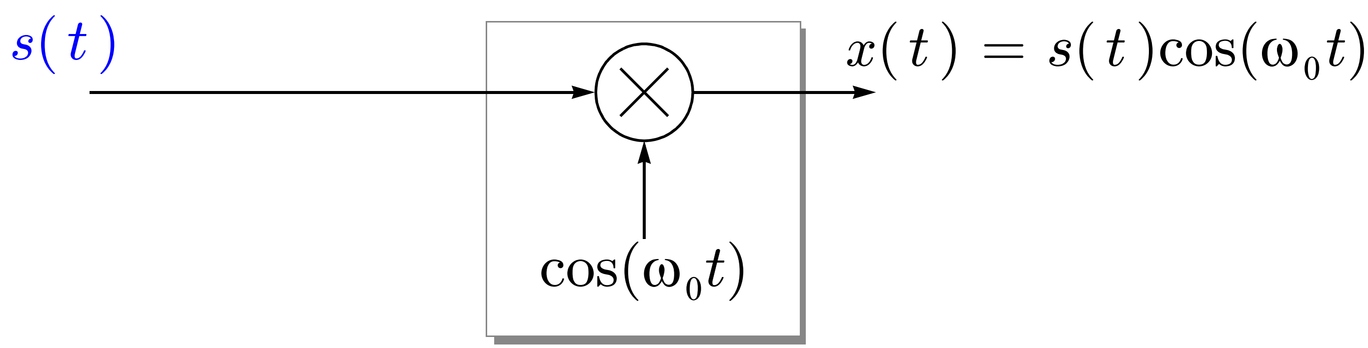
What if Modulating Signal Goes Negative?

Can we recover red signal from black (AM-modulated) signal?



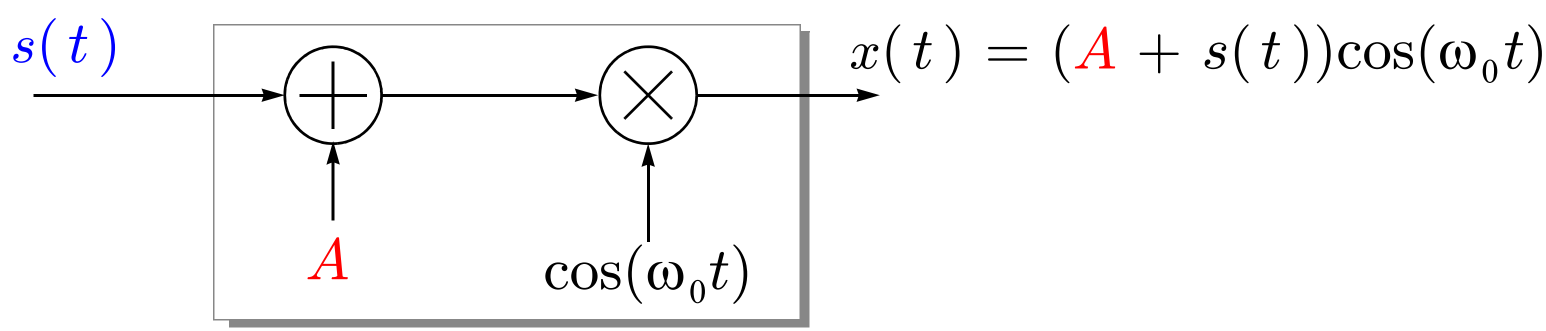
Two Flavors of AM

- AM-DSB-SC (double-sideband, supressed carrier)



  ▷ allow modulating signal to go negative, deal with it at receiver

- AM-DSB-TC (double-sideband, transmit carrier)



  ▷ add a constant to ensure that modulating signal stays positive

  ▷ simplifies receiver

  ▷ at what cost?