Lecture 10: Thu Sep 17, 2020

Reminder:

- HW4 due tonight.
- Coming soon: Quiz 1 is two weeks from today.

Lecture

- review Fourier series
- introduce Fourier transform

Reading Assignment

Handouts for ECE 3084

- syllabus
- 3084 book

6 Review of Fourier Series **37**

6.1 Fourier synthesis sum and analysis integral 37
6.2 System response to a periodic signal 38
6.3 Properties of Fourier series 38
6.4 Fourier series of a symmetric “square 39
6.4.1 Lowpass filtering the square 40
6.5 What makes Fourier series tick? 41
6.6 Under the hood 43

7 Fourier Transforms **45**

7.1 Motivation 45
7.2 A key observation 46
7.3 Your first Fourier transform: decaying exponential 47
7.3.1 Frequency response example 47
7.4 Your first Fourier transform property: time shift 48
7.5 Your second Fourier transform: delta function 48
7.5.1 Sanity check 49
7.6 Your second Fourier transform property: deri… in time 49
7.7 Rectangular boxcar functions 49
7.7.1 Fourier transform of a symmetric rectangular boxcar 49
7.7.2 Inverse Fourier transform of single symmetric boxcar 50
7.7.3 Observations about our boxcar examples 51
7.8 Fourier transforms of deltas and sinusoids 52
7.9 Fourier transform of periodic signals 53

TODAY

NEXT

2025 Reading

LAST WEEK

10 Frequency Response 285

10-1 The Frequency Response Function for LTI Systems . . . 285
10-1.1 Plotting the Frequency Response . . . 287
10-1.1.1 Logarithmic Plot . . . 288
10-1.2 Magnitude and Phase Changes . . . 288
10-2 Response to Real Sinusoidal Signals . . . 289
10-2.1 Cosine Inputs . . . 290
10-2.2 Symmetry of $H(j\omega)$ . . . 290
10-2.3 Response to a General Sum of Sin . . . 293
10-2.4 Periodic Input Signals . . . 294
10-3 Ideal Filters . . . 295
10-3.1 Ideal Delay System . . . 295
10-3.2 Ideal Lowpa . . . 296
10-3.3 Ideal Hi . . . 297
10-3.4 Ideal Banc . . . 297
10-4 Application of Ideal . . . 298
10-5 Time-Domain or Freq Domain? . . . 300
10-6 Summary/Future . . . 301
10-7 Problems . . . 302

NEXT

11 Continuous-Time Fourier Transform 307

11-1 Definition of the Fourier Transform . . . 308
11-2 Fourier Transform and the Spectrum . . . 310
11-2.1 Limit of the Fourier Series . . . 310
11-3 Existence and Convergence of the Fourier Transform . . . 312
11-4 Examples of Fourier Transform Pairs . . . 313
11-4.1 Right-Sided Real Exponential Signals . . . 313
11-4.1.1 Bandwidth and Decay Rate . . . 314
11-4.2 Rectangular Pulse Signals . . . 314
11-4.3 Bandlimited Signals . . . 316
11-4.4 Impulse in Time or Frequency . . . 317
11-4.5 Sinusoids . . . 318
11-4.6 Periodic Signals . . . 319
11-5 Properties of Fourier Transform Pairs . . . 322
11-5.1 The Scaling Property . . . 322
11-5.2 Symmetry Properties of Fourier Transfo . . . 324
11-6 The Convolution Property . . . 326
11-6.1 Frequency Response . . . 326
11-6.2 Fourier Transform of a Convolution . . . 327
11-6.3 Examples of the Use of the Convolution Property . . . 328
11-6.3.1 Convolution of Two Bandlimited Functions . . . 328
11-6.3.2 Product of Two Sinc Functions . . . 329
11-6.3.3 Partial Fraction Expansions . . . 330
11-7 Basic LTI Systems . . . 332
11-7.1 Time Delay . . . 332
11-7.2 Differentiation . . . 333
11-7.3 Systems Described by Differential Equations . . . 334
11-8 The Multiplication Property . . . 335
11-8.1 The General Signal Multiplication Property . . . 335
11-8.2 The Frequency Shifting Property . . . 336
11-9 Table of Fourier Transform Properties and Pairs . . . 337
11-10 Using the Fourier Transform for Multipath Analysis . . . 337
11-11 Summary . . . 341
11-12 Problems . . . 342

TODAY

3 Spectrum Representation 36

3-1 The Spectrum of a Sum of Sinusoids . . . 36
3-1.1 Notation Change . . . 38
3-1.2 Graphical Plot of the Spectrum . . . 38
3-2 Beat Notes . . . 39
3-2.1 Multiplication of Sinusoids . . . 39
3-2.2 Beat Note Waveform . . . 40
3-2.3 Amplitude Modulation . . . 41
3-3 Periodic Waveforms . . . 43
3-3.1 Synthetic Vo . . . 44
3-3.2 Examp . . . 45
3-4 Fourier Serie . . . 47
3-4.1 Fourier . . . 48
3-4.2 Fourier S ion . . . 48
3-5 Spectrum of the Fo Series . . . 50
3-6 Fourier Analysis of Periodic Signals . . . 51
3-6.1 The Square Wave . . . 52
3-6.1.1 DC Value of a Square Wave . . . 53
3-6.2 Spectrum for a Square Wave . . . 53
3-6.3 Synthesis of a Square Wave . . . 54
3-6.4 Triangle Wave . . . 55
3-6.5 Synthesis of a Triangle Wave . . . 56
3-6.6 Convergence of Fourier Synthesis . . . 57
3-7 Time–Frequency Spectrum . . . 57
3-7.1 Stepped Frequency . . . 59
3-7.2 Spectrogram Analysis . . . 59
3-8 Frequency Modulation: Chirp Signals . . . 60
3-8.1 Chirp or Linearly Swept Frequency . . . 60
3-8.2 A Closer Look at Instantaneous Frequency . . . 62
3-9 Summary and Links . . . 63
3-10 Problems . . . 64

Periodic Signals

A periodic signal satisfies $x(t) = x(t + T)$ for all t.

The smallest nonzero T that works is the *fundamental* period.

Key fact from 2026:

Any (!) periodic signal can be written as a sum of sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T} \qquad \text{"FS synthesis"}$$

with *harmonically* related frequencies, whose amplitudes and phases are determined by the Fourier series coefficients:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk2\pi t/T} dt \qquad \text{"FS analysis"}$$

Any interval of length T,
e.g. $[0, T)$, $[-T/2, T/2)$, etc.

Implications of Even or Odd

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk2\pi t/T}dt$$

- If $x(t)$ is even, then a_k are ...
- If $x(t)$ is odd, a_k are ...

Implications of Even or Odd

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk2\pi t/T}dt$$

$$= \frac{1}{T}\int_T x(t)\big(\cos(k2\pi t/T) - j\sin(k2\pi t/T)\big)dt$$

- If $x(t)$ is even, then a_k are ...
- If $x(t)$ is odd, a_k are ...

Implications of Even or Odd

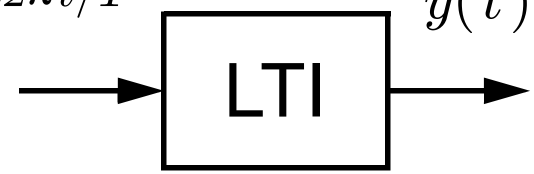
$$a_k = \frac{1}{T}\int_T x(t) e^{-jk2\pi t/T} dt$$

$$= \frac{1}{T}\int_T x(t)\big(\cos(k2\pi t/T) - j\sin(k2\pi t/T)\big) dt$$

- If $x(t)$ is even, then a_k are real and even.
- If $x(t)$ is odd, a_k are purely imaginary.
  E.g., consider FS for $\sin(200\pi t)$.

Filtering a Periodic Signal

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$



$y(t) = ?$

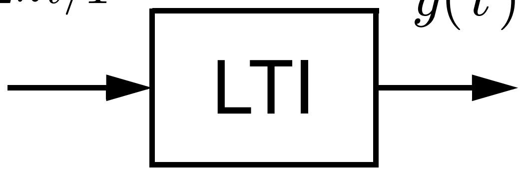
Pop Quiz (True or False):

(a) Output is always periodic.

(b) Output fundamental period is the same.

Filtering a Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$



$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk2\pi/T) e^{jk2\pi t/T}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk2\pi t/T}$$

Pop Quiz (True or False):

(a) Output is always periodic.

(b) Output fundamental period is the same.

Consider cos(80Hz)+cos(90Hz) after LPF(85 Hz).)

More Fun FS Facts

- Linearity: When both have same period, FS coeffs of $\alpha x(t) + \beta y(t)$ are generally $\alpha a_k + \beta b_k$

(Caveat: unless period changes!
e.g., $x(t) = 10$ Hz $+$ 20 Hz, $y(t) = -10$ Hz $+$ 40 Hz)

- Filter by $h(t) \Rightarrow$ multiply a_k by $H(jk2\pi/T)$

- Delay by $t_0 \Rightarrow$ multiply a_k by $e^{-jk2\pi t_0/T}$

- Differentiate $\Rightarrow$ multiply a_k by $jk2\pi/T$

- *(Parseval's relationship):* The power of a periodic signal is:

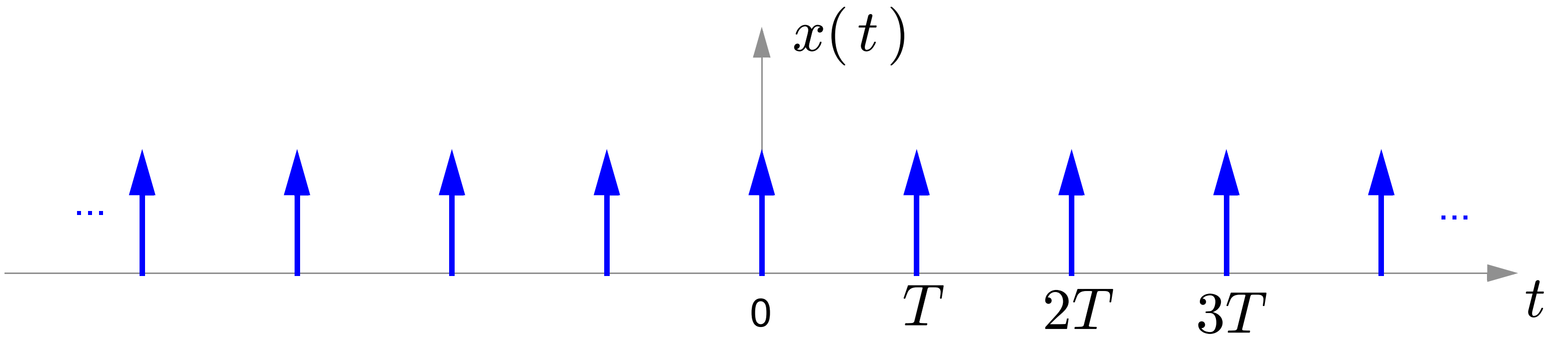
$$P = \frac{1}{T}\int_T x^2(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Parseval's Relationship

The power of a periodic signal is:

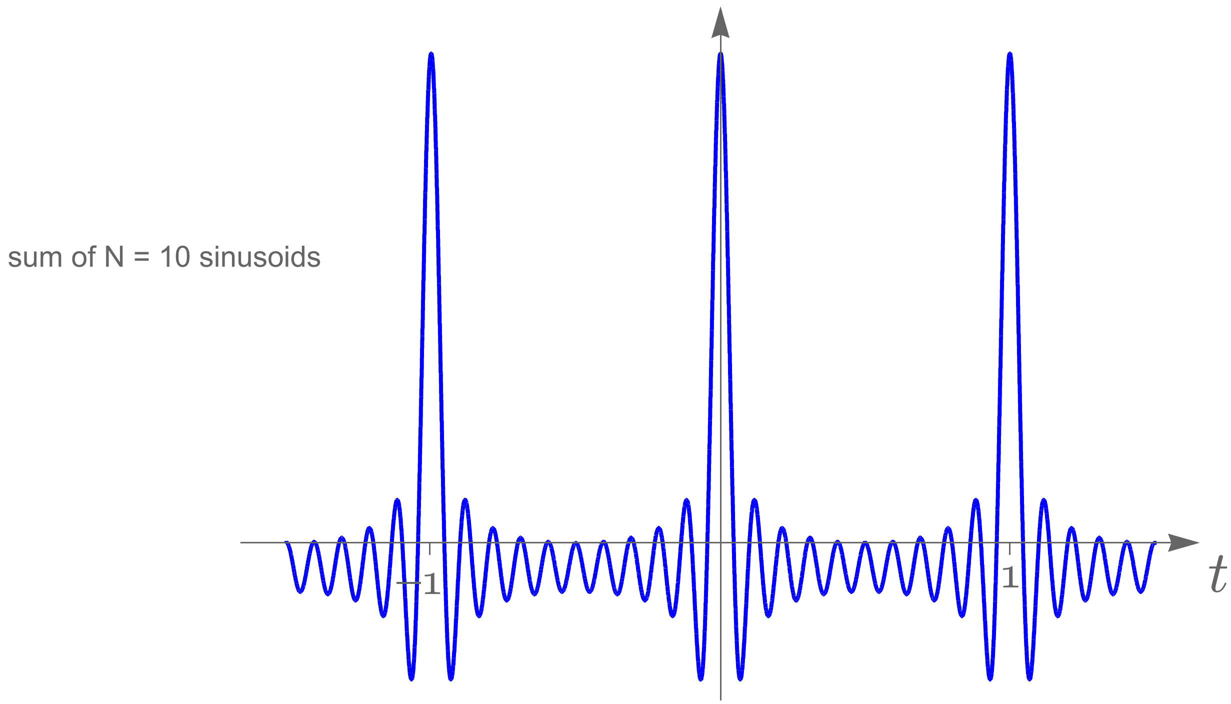
$$\frac{1}{T}\int_T x^2(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty}|a_k|^2$$

Example: Pulse Train



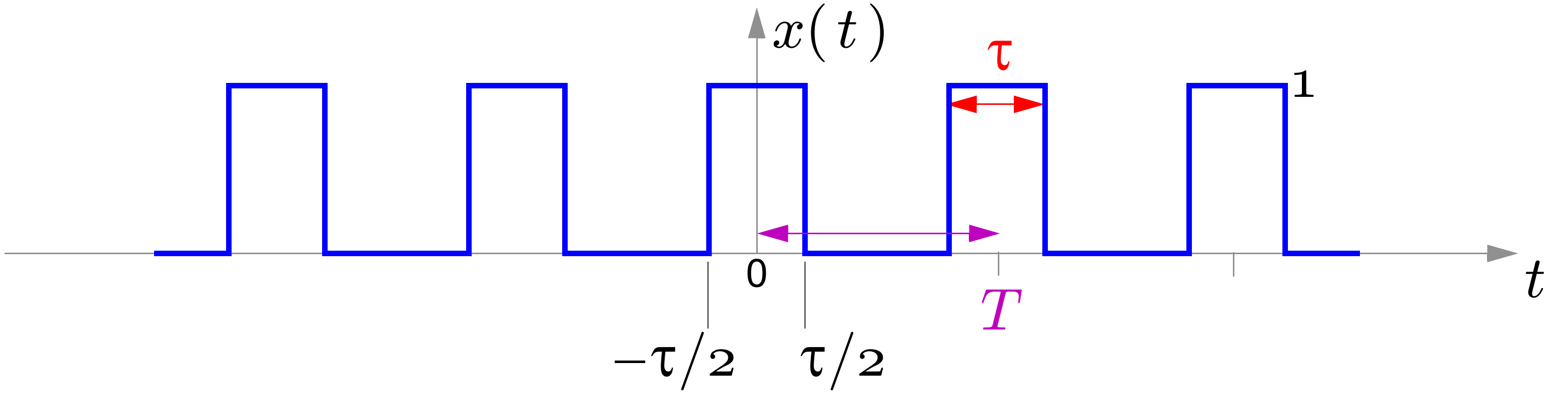
What are Fourier series coeffs?

MATLAB Demo



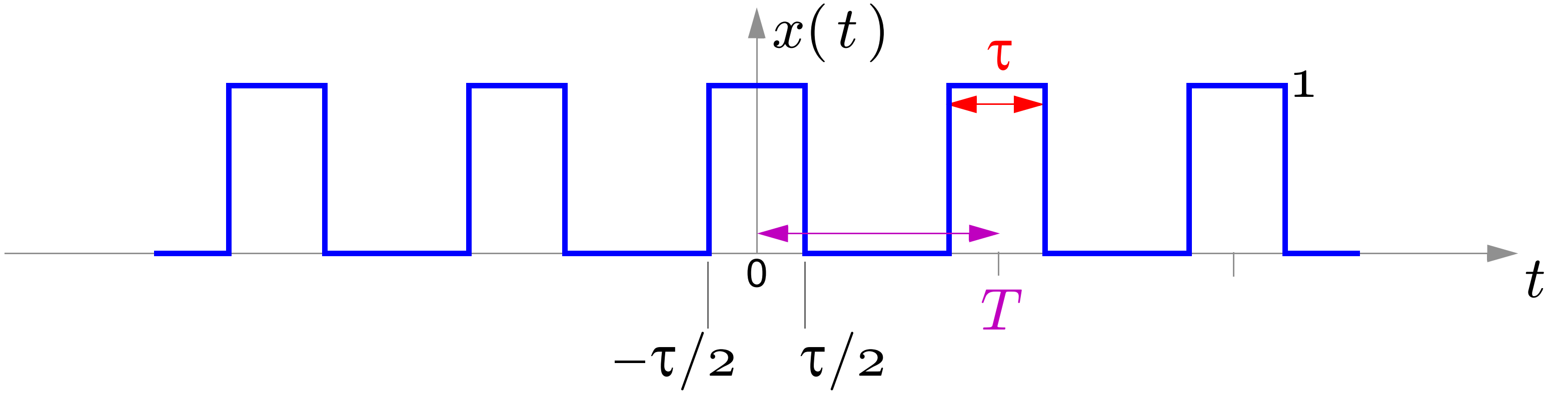

```
t = -1.5:1e-4:1.5;
for N = [1 2 3 5 10 100 1e3 1e4],
   x = 0;
   for k=1:N, x = x + cos(2*pi*k*t); end;
   plot(t,x);
   title(['sum of N = ',num2str(N),' sinusoids']);
   input('Hit return to continue ','s');
end
```

Example 1: Duty-Cycle = τ/T



Prediction: $x(t)$ is even $\Rightarrow a_k$ is ...

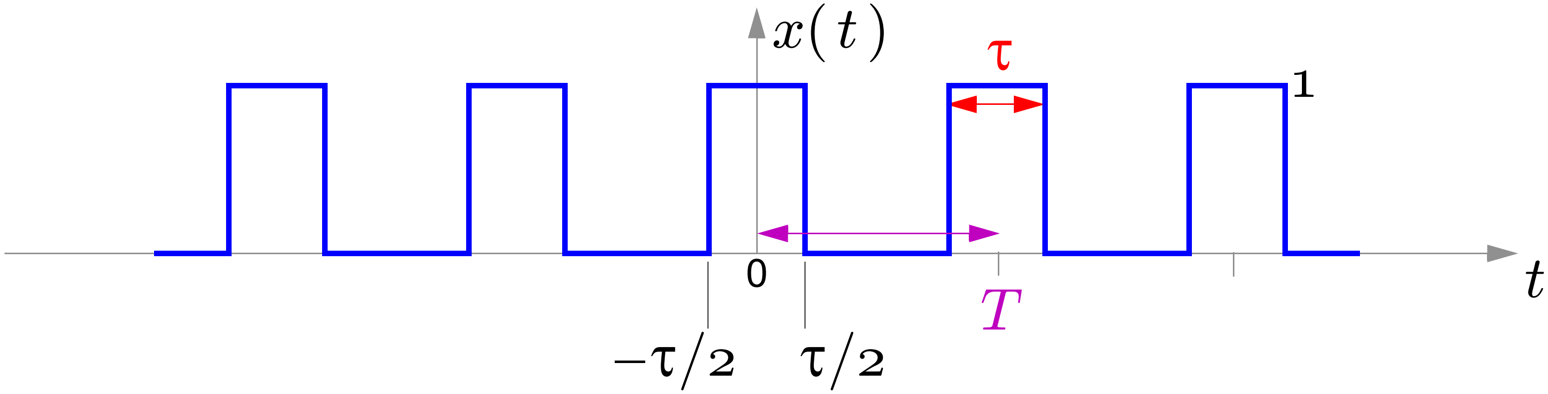
Example 1: Duty-Cycle = τ/T



Prediction: $x(t)$ is even $\Rightarrow a_k$ purely real.

Compute a_0 separately: $a_0 =$

Example 1: Duty-Cycle = τ/T



Prediction: $x(t)$ is even $\Rightarrow a_k$ purely real.

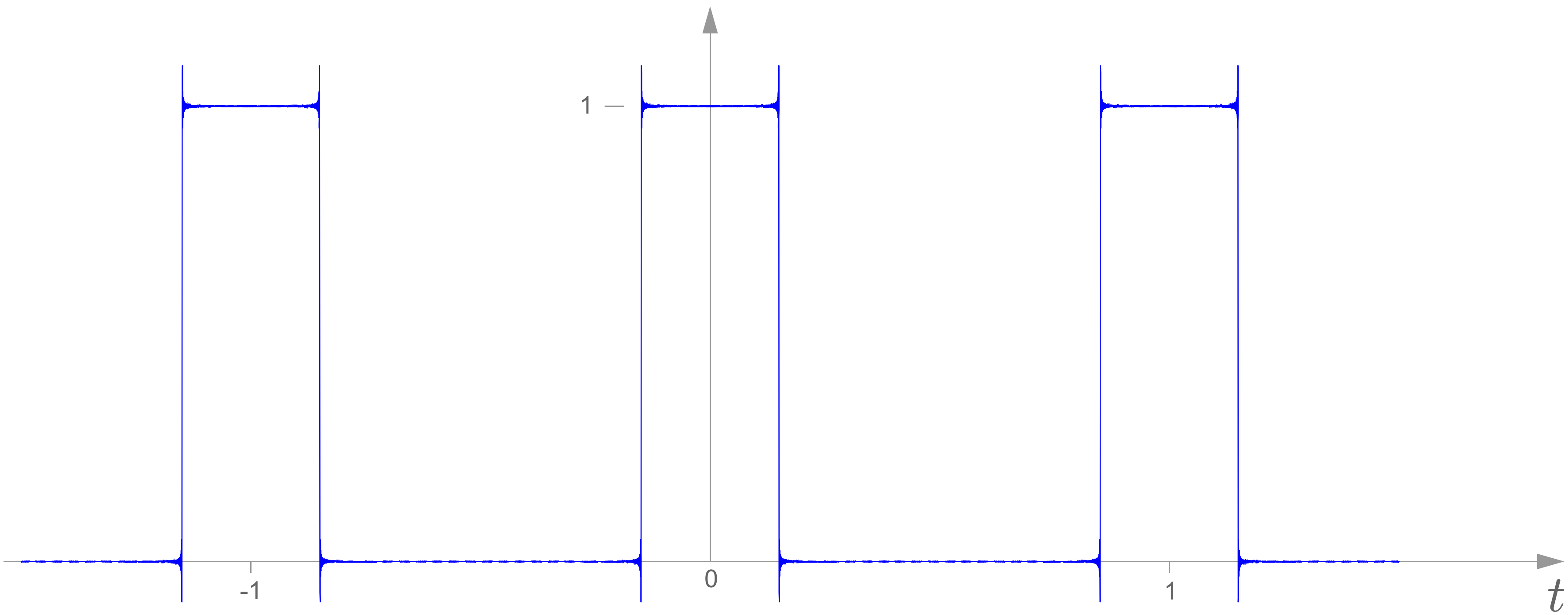
Compute a_0 separately: $a_0 = \frac{\tau}{T}$

Compute the rest (for $k \neq 0$):

$$a_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-jk2\pi t/T} dt = \left.\frac{e^{-jk2\pi t/T}}{-jk2\pi}\right|_{-\tau/2}^{\tau/2} = \frac{\sin(k\pi\tau/T)}{k\pi}$$

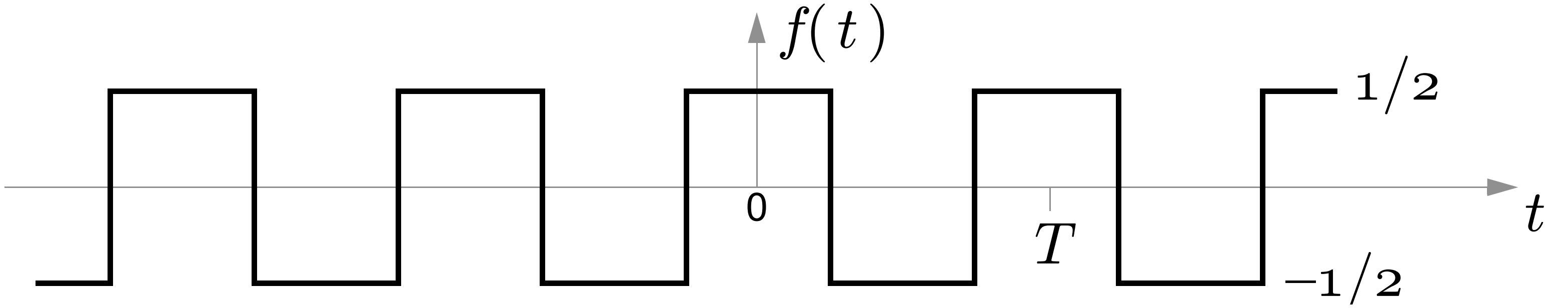
$$= \frac{\tau}{T}\text{sinc}(k\frac{\tau}{T}).$$

Example 1: 30% Duty Cycle




```
t = -1.5:1e-4:1.5;
duty = 0.3;
for N = [1 2 3 5 10 100 1e3 1e4],
      x = 0;
      for k=-N:N,
            ak = 0.3*sinc(0.3*k);
            x = x + ak*exp(2i*pi*k*t);
      end;
      plot(t,x);
      title(['sum of N = ',num2str(N),' sinusoids']);
      input('Hit return to continue ','s');
end
```

Example 2: 50% duty cycle, No DC

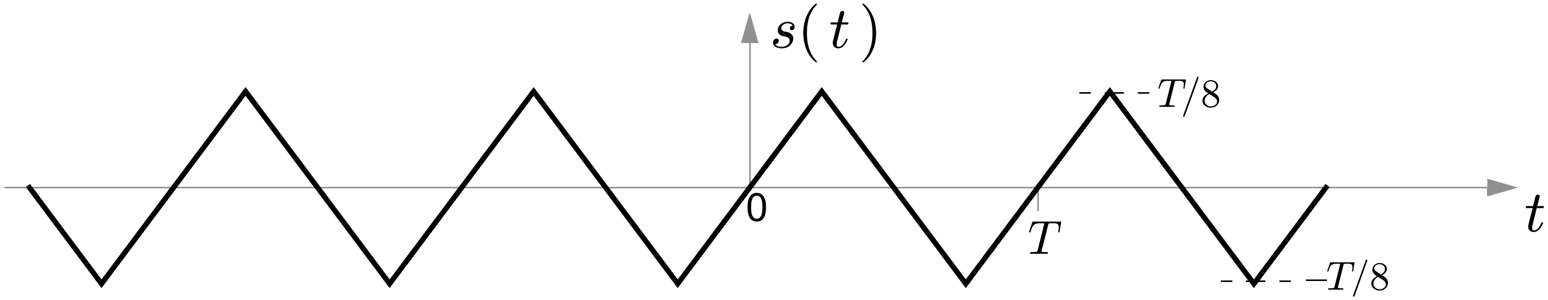


$f(t) = x(t) - 1/2$

$\Rightarrow$ only a_0 changes:

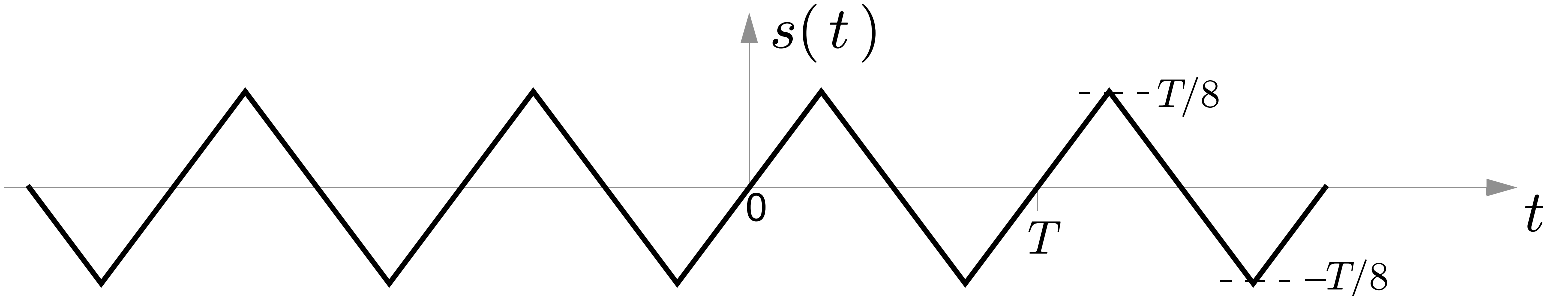
$$\Rightarrow f(t) = \sum_{k \neq 0} a_k e^{jk2\pi t/T}$$

Example 3: SAWTOOTH



Prediction: $s(t)$ is odd $\Rightarrow$ its FS coeffs c_k will be ...

Example 3: SAWTOOTH



Prediction: $s(t)$ is odd $\Rightarrow$ its FS coeffs c_k will be purely *imaginary*

Let b_k be FS coeffs of $f(t)$ (previous page)

Previous signal is derivative of this one!

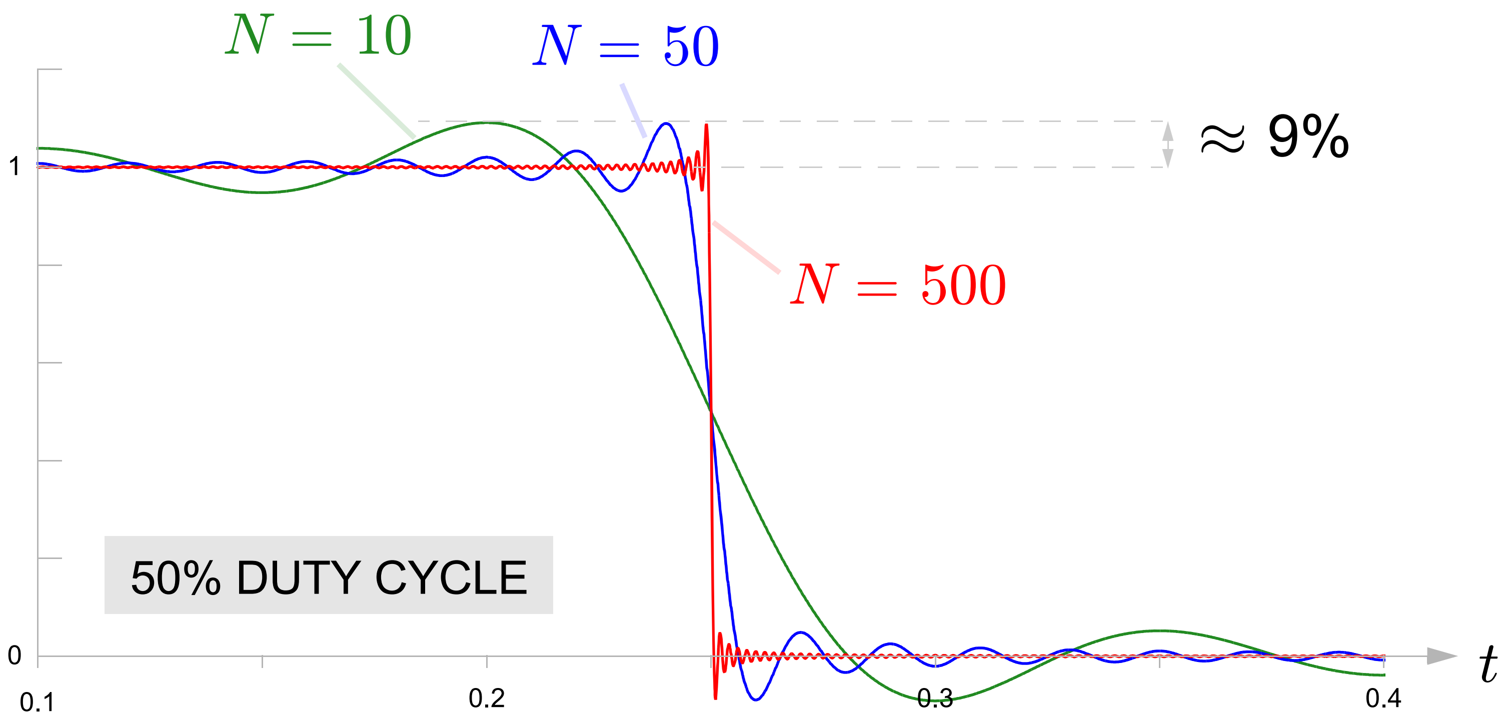
$$\Rightarrow f(t) = \frac{d}{dt} s(t)$$

$$\Rightarrow b_k = (jk2\pi/T)c_k$$

$$\Rightarrow c_k = \frac{b_k}{jk2\pi/T} = \frac{\tau \mathrm{sinc}(k\tau/T)}{jk2\pi} = \frac{\sin(k\pi\tau/T)}{j2(k\pi)^2/T} \quad \text{for } k \neq 0,\ c_0 = 0$$

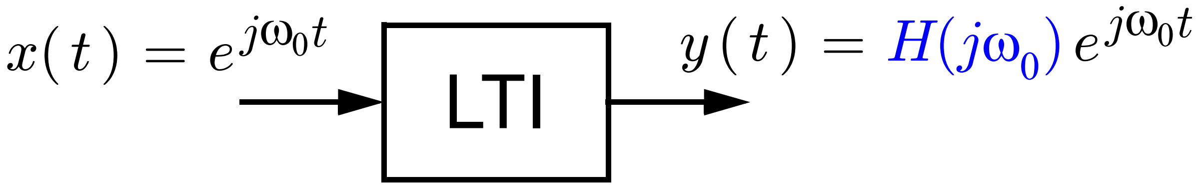
Gibbs Phenom

If periodic $x(t)$ has discontinuities, the *finite* approx $\sum_{k=-N}^{N} a_k e^{jk2\pi t/T}$ exhibits ringing and overshoot at discontinuity:



- *height* of overshoot stays at about 9%, regardless of N
- *width* of ringing diminishes as N increases

Recall: Sinusoid-In, Sinusoid Out



Where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$ “frequency response”

What’s different now?

▹ apply this integral to *any* signal, not just impulse response:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ “Fourier transform” of signal $x(t)$

▹ analogous to DTFT from 2026. Only difference: here CT, not DT.