Step and Delta Functions
18.031
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1 The unit step function

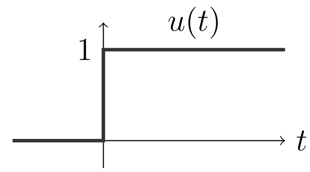
1.1 Definition

Let's start with the definition of the unit step function, $u(t)$:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

We do not define $u(t)$ at $t = 0$. Rather, at $t = 0$ we think of it as in transition between 0 and 1.

It is called the unit step function because it takes a unit step at $t = 0$. It is sometimes called the Heaviside function. The graph of $u(t)$ is simple.



We will use $u(t)$ as an idealized model of a natural system that goes from 0 to 1 very quickly. In reality it will make a smooth transition, such as the following.

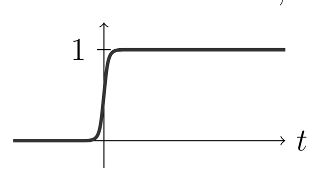


Figure 1. $u(t)$ is an idealized version of this curve

But, if the transition happens on a time scale much smaller than the time scale of the phenomenon we care about then the function $u(t)$ is a good approximation. It is also much easier to deal with mathematically.

One of our main uses for $u(t)$ will be as a switch. It is clear that multiplying a function $f(t)$ by $u(t)$ gives

$$u(t)f(t) = \begin{cases} 0 & \text{for } t < 0 \\ f(t) & \text{for } t > 0. \end{cases}$$

We say the effect of multiplying by $u(t)$ is that for $t < 0$ the function $f(t)$ is switched off and for $t > 0$ it is switched on.

1.2 Integrals of $u'(t)$

From calculus we know that

$$\int u'(t)\,dt = u(t) + c \quad \text{and} \quad \int_a^b u'(t)\,dt = u(b) - u(a).$$

For example:

$$\begin{aligned}
\int_{-2}^{5} u'(t)\,dt &= u(5) - u(-2) = 1, \\
\int_{1}^{3} u'(t)\,dt &= u(3) - u(1) = 0, \\
\int_{-5}^{-3} u'(t)\,dt &= u(-3) - u(-5) = 0.
\end{aligned}$$

In fact, the following rule for the integral of $u'(t)$ over any interval is obvious

$$\int_a^b u'(t) = \begin{cases} 1 & \text{if 0 is inside the interval } (a,b) \\ 0 & \text{if 0 is outside the interval } [a,b]. \end{cases} \tag{1}$$

Note: If one of the limits is 0 we throw up our hands and refuse to do the integration.

Let 0^- be infinitesimally to the left of 0 and 0^+ infinitesimally to the right of 0. That is,

$$0^- < 0 < 0^+.$$

For a function, $f(0^-)$ is defined as the left hand limit at 0 or equivalently the limit from below at 0, provided this limit exists. Likewise $f(0^+)$ is the right hand limit or the limit from above.

$$f(0^-) = \lim_{t \uparrow 0} f(t) \qquad f(0^+) = \lim_{t \downarrow 0} f(t)$$

Here are some examples of integrals of u' that involve 0^- and 0^+:

$$\begin{aligned}
\int_{-\infty}^{0^+} u'(t)\,dt &= 1 \quad (\text{because } -\infty < 0 < 0^+), \\
\int_{-\infty}^{0^-} u'(t)\,dt &= 0 \quad (\text{because } -\infty < 0^- < 0), \\
\int_{0^-}^{0^+} u'(t)\,dt &= 1 \quad (\text{because } 0^- < 0 < 0^+).
\end{aligned}$$

1.3 Preview of generalized functions and derivatives

Of course $u(t)$ is not a continuous function, so in the 18.01 sense its derivative at $t = 0$ does not exist. Nonetheless we saw that we could make sense of the integrals of $u'(t)$. So rather than throw it away we call $u'(t)$ the generalized derivative of $u(t)$. You can't do everything with $u'(t)$ you can do with an ordinary function, but it can go anywhere we have an input function in 18.03. In the next section we will look in more detail at $u'(t)$ –and call it $\delta(t)$. For now we'll content ourselves with computing the Laplace transform of u and u'.

1.4 The Laplace transform of $u(t)$ and $u'(t)$

This is easy since $u(t)$ is identical to the constant function 1 on the interval $(0, \infty)$ of the Laplace transform. Therefore

$$\mathcal{L}(u(t)) = 1/s.$$

(We could also compute this directly from the definition $\mathcal{L}(u) = \int_{0^-}^{\infty} u(t)e^{-st}\,dt$.)

For u', we use the formula $\mathcal{L}(u') = s\mathcal{L}(u) - u(0^-)$ and the fact that $u(0^-) = 0$ to get

$$\mathcal{L}(u') = s \cdot \frac{1}{s} = 1.$$

1.5 The unit step response

Suppose we have an LTI system with system function $H(s)$. The unit step response of this system is defined as its response to input $u(t)$ with rest initial conditions.

Theorem. The Laplace transform of the unit step response is $H(s)\,\frac{1}{s}$.

Proof. This is a triviality since in the frequency domain: output = transfer function $\times$ input.

Example 1. Consider the system $\dot{x} + 2x = f(t)$, with input f and response x. Find the unit step response.

answer: We have $f(t) = u(t)$ and rest initial conditions. The system function is $1/(s + 2)$, so by the theorem, the unit step response written in terms of frequency is given by

$$X(s) = \frac{1}{s(s+2)}$$

The partial fractions decomposition is $X(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$, so in the time domain the unit step response is $x(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$ for $t > 0$. (Of course $x(t) = 0$ for $t < 0$.)

Example 2. In the previous example, find the long-term behavior of the unit step response in two ways.

answer: Method 1: Compute the limit directly.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1}{2} - \frac{1}{2}e^{-2t} = \frac{1}{2}.$$

Method 2: Use the final value theorem. (If you haven't covered that in class just skip this method –or go back and read about the final value theorem in the reading on Laplace transform.) We have $sX(s) = 1/(s+2)$. Since all its poles are negative, we can apply the final value theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \frac{1}{2}.$$

We see that both methods agree!

2 The unit impulse

In this section we will learn about the unit impulse function $\delta(t)$. We will use it as input to LTI systems. At first the systems will be simple enough to find the post-initial conditions directly and use them to solve the equations for the response. For more complicated systems we will use the Laplace transform to solve the equation without first determining the post-initial conditions.

2.1 The mathematics of the delta function

Let's delve a little deeper into $u'(t)$. It's clear $u'(t) = 0$ if $t \neq 0$. At $t = 0$ the curve is vertical so the slope is infinite, i.e. $u'(0) = \infty$. (If you think of $u(t)$ as an idealized version of the curve in Figure 1, then we would say the derivative near 0 gets very large.) We define

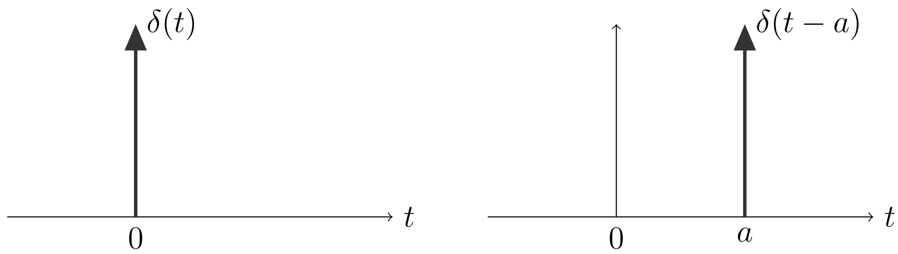
$$\delta(t) = u'(t)$$

and call it the delta function or the Dirac delta function or the unit impulse function. We have seen the following properties of $\delta(t)$.

1. $\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0. \end{cases}$

2. $\displaystyle\int \delta(t)\, dt = u(t)$ and $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$

Based on property 1, we 'graph' $\delta(t)$ as an infinite spike at the origin. The integrals show that the 'area' under this graph equals 1 and it is all concentrated at the origin.



We also show $\delta(t-a)$ which is just $\delta(t)$ shifted to the right.

2.2 The non-idealized delta function

Just like the unit step function, the δ function is really an idealized view of nature. In reality, a delta function is nearly a spike near 0 which goes up and down on a time interval much smaller than the scale we are working on. The integral, i.e. area under the curve, is always 1. It's graph might actually look something like

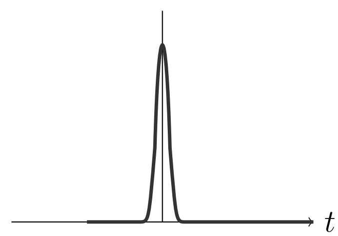


Figure 2. Non-idealized delta function; area under the graph = 1.

The total amount input is still the integral (see Section 2.4 below), or, in geometric terms, the area under the graph. For a unit impulse we assume the area is 1.

2.3 Delta functions are your friend

2.3.1 Integrals with $\delta(t)$

Recall how painful integration could be. In contrast, integrals with delta functions are always easy and involve no techniques of integration.

Suppose we scale $\delta(t)$: the integrals are just scaled.

$$\int_{-5}^{5} 3\delta(t)\,dt = 3, \quad \int_{-5}^{-3} 3\delta(t)\,dt = 0, \quad \int_{0^-}^{0^+} 3\delta(t)\,dt = 3, \quad \int_{0^+}^{\infty} 3\delta(t)\,dt = 0.$$

The integral $\int_a^b f(t)\delta(t)\,dt$ is also easy. If $f(t)$ is continuous at $t=0$ then

$$\int_a^b f(t)\delta(t)\,dt = \begin{cases} f(0) & \text{if } (a,b) \text{ contains } 0 \\ 0 & \text{if } [a,b] \text{ does not contain } 0. \end{cases}$$

That is, integrating against $\delta(t)$ just amounts to evaluating $f(t)$ at $t=0$.

Note 1. If one of the endpoints a or b is 0, the integral cannot be evaluated, so we just throw up our hands and refuse to do it.

Note 2. Technicality: We must have $f(t)$ continuous at $t = 0$.

2.3.2 Justification of the formula for $\int f(t)\delta(t)\,dt$

We should start by admitting that in formal mathematics this is simply given as the definition of $\delta(t)$, so our arguments will just go to show that it is a reasonable definition. We'll do this in three ways.

Quick reason: $\delta(t)$ is 0 everywhere except $t = 0$, So $f(t)\delta(t)$ is 0 for all $t \neq 0$ and at $t = 0$ it just scales the delta function by $f(0)$. That is, $f(t)\delta(t) = f(0)\delta(t)$.

Reason 1. Since we can interpret the integral as area, we need to show that the 'area' under $f(t)\delta(t)$ is $f(0)$. Figure 2 (above) shows a tall, thin curve near $t = 0$ which approximates $\delta(t)$. Since $f(t)$ is continuous we know that $f(t) \approx f(0)$ near $t = 0$. Thus, $f(t)\delta(t)$ is approximated by the graph in the figure scaled by $f(0)$. Finally, since the area under the curve in Figure 2 is one, if we scale it by $f(0)$ it will have area equal to $f(0)$. As the graph in Figure 2 gets narrower and taller it goes to that of $\delta(t)$. As this happens, the approximation we just made will become exact, i.e. as we wanted to show, the area under the $f(t)\delta(t) = f(0)$.

Reason 2. This is a direct argument using integration by parts. First, since $\delta(t) = 0$ for $t \neq 0$ the integral $\int_a^b f(t)\delta(t)\,dt$ must be zero for any interval $[a, b]$ not containing 0. Next, suppose $a < 0 < b$, then we get

$$\begin{aligned}\int_a^b f(t)\delta(t)\,dt &= \int_a^b f(t)u'(t)\,dt \quad (\text{since } \delta = u') \\ &= f(t)u(t)|_a^b - \int_a^b f'(t)u(t)\,dt \quad (\text{integration by parts})\end{aligned}$$

Now, since $u(b) = 1$, $u(a) = 0$ and $u(t) = 0$ for $t < 0$ this becomes

$$\begin{aligned} &= f(b) - \int_0^b f'(t)\,dt \\ &= f(b) - f(t)|_0^b \\ &= f(b) - f(b) + f(0) \\ &= f(0)\end{aligned}$$

Comparing the first and last expressions in this long sequence of steps, we've shown the result.

2.3.3 Shifting by a

If we shift by a we have, $\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)$. More generally:

$$\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & \text{if } (c,d) \text{ contains a} \\ 0 & \text{if } [c,d] \text{ does not contain a.} \end{cases}$$

Example 3. (Practice with δ.) Quickly cover up the answers on the left and try to evaluate each of the integrals on the right.

$$\begin{aligned}
&\int_{-1}^{3} \delta(t)2e^{4t^2}\,dt && = 2, && \text{(evaluate } 2e^{4t^2} \text{ at } t=0) \\
&\int_{1}^{3} \delta(t)2e^{4t^2}\,dt && = 0, && \text{(0 is not in [1,3])} \\
&\int_{0^-}^{3} \delta(t)2e^{4t^2}\,dt && = 2, && \text{(evaluate } 2e^{4t^2} \text{ at } t=0) \\
&\int_{0^-}^{\infty} \delta(t)2e^{-\tan^2(t^3)}\,dt && = 2, && \text{(evaluate } 2e^{-\tan^2(t^3)} \text{ at } t=0) \\
&\int_{-1}^{3} \delta(t-2)2e^{4t^2}\,dt && = 2e^{16}, && \text{(evaluate } 2e^{2e^{4t^2}} \text{ at } t=2) \\
&\int_{3}^{5} \delta(t-2)2e^{4t^2}\,dt && = 0, && \text{(2 is not in [3,5])} \\
&\int_{0^-}^{3} \delta(t-2)2e^{4t^2}\,dt && = 2e^{16} && \text{(evaluate } 2e^{2e^{4t^2}} \text{ at } t=2), \\
&\int_{0^-}^{\infty} \delta(t-2)2e^{-\tan^2(t^3)}\,dt && = 2e^{-\tan^2(8)} && \text{(evaluate } 2e^{-\tan^2(t^3)} \text{ at } t=2).
\end{aligned}$$

2.4 The physical interpretation of delta functions as a unit impulse

In general, we will be using δ functions as the input to LTI systems. So, in this subsection we want to explore what this means. Our goal is to understand what is meant by an impulse and to see that $\delta(t)$ can be thought of as an (idealized) unit impulse.

Example 4. Consider the rate equation $\dot{x} + kx = f(t)$. To be specific, assume x is in units of kilograms and t is in minutes. This is a rate equation and the derivative $\dot{x}$ and the input $f(t)$ are rates, in units of kg/min. We then have that the total amount of kg input from time 0^- to time t is $\int_{0^-}^{t} f(\tau)\,d\tau$.

Consider the following possible inputs $f(t)$, shown graphically as box functions.