Signals and Systems for *All* Electrical Engineers

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August 17, 2020

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Change Log

This document is a rough pre-alpha draft; please **do not redistribute.** It will undergo constant revision.

- August 17, 2020: Change Log rebooted.

Preface

Dozens, if not hundreds, of electrical engineering textbooks have been written with phrase "Signals and Systems" or "Systems and Signals" in the title. Occasionally their titles will also contain the words "Circuits" or "Transforms," and recent years, it has become fashionable to add "with MATLAB" somewhere on the cover. These texts are commonly used for junior-level classes, although there are now many examples of departments teaching this material at the sophomore level.

We believe this junior-level text, which focuses heavily on continuous-time signals and systems, is unique in that it is intended to follow sophomore-level courses on discrete-time signals and systems and linear electrical circuits.

0.1 The *DSP First* Legacy

Although courses sophomore-level courses on linear circuits are ubiquitous, sophomore-level courses on discrete-time signals and systems are comparatively rare.

In the early 1990s, Georgia Tech professors Ron Schafer, Jim McClellan, and Tom Barnwell started to explore using digital signal processing, instead of electric circuits, as an introduction to electrical engineering. Schafer and McClellan joined with Rose-Hullman professor Mark Yoder to pen *DSP First*, which formed the basis of a quarter-length course at Georgia Tech. The "signal processing" aspect was somewhat incidental; at its core, the course was an introduction to signal and system concepts that happened to use DSP, particularly discrete-time filtering, as a compelling application. The course began with continuous-time sinusoids, their complex phasor representations, and Fourier series, but quickly sampled those sinusoids to explore frequency response concepts the discrete-time context.

Although it may seem unusual at first glance, the discrete-time domain offers several pedagogical advantages over the continuous-time domain. Sampling and aliasing are somewhat counterintuitive; even today, well-known analog engineers occasionally make statements that betray a profound misunderstanding of the Nyquist sampling theorem. Hence, we have found it useful to introduce sampling early in the curriculum. Also, mathematical complications in continuous-time theory, such as the generalized function aspects of the Dirac delta function, are avoided in a discrete-time context, which may make the discrete-time theory slightly easier to explain at the sophomore level. However, the strongest advantage of the *DSP First* approach is that allow students to readily put theory into practice using appropriate tools such as MATLAB. The continuous-time theory is often more abstract, and appropriate lab exercises require specialized equipment; in contrast, discrete-time signal processing can be done with a common personal computer. Hence, a formal lab section with concrete audio and image processing examples in MATLAB was an integral part of the course.

Shortly after *DSP First* was published, the Board of Regents of the University System of Georgia forced a quarter-to-semester conversion upon Georgia Tech. The *DSP First* authors found themselves with a text that covered only two-thirds of a semester. They decided to write additional chapters that covered continuous-time system theory, largely covering concepts in the same order that they were covered in the discrete-time portion of the course. Since students had already seen the concepts in the discrete-time domain, they were able to cover the continuous-time concepts efficiently. Splitting the material into a discrete-time part followed by a continuous-time part was opposite the order of many textbooks, in which the early chapters cover the continuous-time domain and later chapters cover the discrete-time domain. It also contrasted with textbooks that alternate domains, covering the continuous-time and discrete-time renditions of each concept in close succession.

The resulting four-credit-hour semester course, ECE2025, spawned the textbook *Signal Processing First* and became a signature course of Georgia Tech's School of Electrical and Computer Engineering. In spite of it being a four-credit-hour course, it felt incomplete. Although we covered discrete-time and continuous-time convolution, and discrete-time and continuous-time frequency responses, there was not enough time in the semester to cover Laplace transforms as the natural analogue[1] to the previously covered z-transforms.

At the time, ECE2040: Circuit Analysis listed ECE2025 as a prerequisite. After being introduced to sinusoids, complex exponential notation, and frequency concepts in ECE2025, students could tackle circuit theory with a stronger conceptual background. Georgia Tech professors Russ Mersereau and Joel Jackson emphasized this connection in their textbook *Circuit Analysis: A Systems Approach*, which could colloquially be though of as "Circuit Analysis Second" as a follow-on to *DSP First* or *Signal Processing First*. They placed a strong emphasis on the Laplace transform, rounding out an aspect of ECE2025 that felt incomplete. Unfortunately, some of the faculty that traditionally taught ECE2040 never fully embraced this approach, and continued teaching ECE2040 exactly the way they might have during the Nixon or Carter administrations.

0.2 Forward to the Past

In 2010, when the School of ECE at Georgia Tech undertook an evaluation and revision of its undergraduate curriculum, many alumni suggested that ECE2025 should be left as it was. However, experience had shown that many students had trouble keeping up with the pace of ECE2025. In particular, some students seemed to get lost shortly after the transition to continuous-time. At that point, the labs, which explored deeper applications in the discrete-time context, started to feel decoupled from the more abstract continuous-time lecture material. Proposals arose to split the material in ECE2025 into two courses. Around the same time, pressure mounted to decouple ECE2040 from ECE2025, formally returning ECE2040 to its default disco-era state.

The result of these negotiations was that ECE2025 would be split into ECE2026: Introduction to Signal Processing, which would now more closely resemble the original quarter-length course from the 1990s, and ECE3084: Signals and Systems. The original discrete-time material from the first two-thirds of ECE2025 would remain in ECE2026 (with some material on the discrete Fourier transform added to ECE2026 to round out the course, and the overall pace of delivery made less intense), and the continuous-time material from the last third of ECE2025 would be moved to ECE3084. Hence, the way we cover that material, and the order in which we cover it, is strongly influenced by *Signal Processing First*. We review Fourier series in Chapter 6, even though Fourier series are still covered in ECE2026, since students find Fourier series to be one of the most challenging topics in ECE2026, and Fourier series are a natural path for introducing Fourier transforms in Chapter 7. ECE3084 then follows up where the original ECE2025 left off, with coverage of Laplace transforms and feedback, with emphasis on the application of feedback in circuit design and control

[1] Pun vaguely intended.

systems, particularly proportional-integral-derivative (PID) control. Much emphasis is placed on the time-domain and frequency-domain responses of second-order systems, since a thorough understanding of such behavior is key to understanding higher-order systems. Links between ECE3084 and ECE2026 are made in Chapter 9, where sampling, which is addressed in a simplified form in ECE2026 for cases of single sinusoids, is revisited from the standpoint of Fourier theory, and Chapter 12, where the z-transforms that play a central role in ECE2026 are shown to be just a special case of Laplace transforms with some convenient, customized notation.

0.3 Putting Educational Eggs in Many Baskets – Not Just One

Occasionally, it seems like students promptly forget almost everything they learned in a class the moment they turn in their final exam. We have found that students benefit from seeing the same topic explored from different perspectives in different classes, particularly in different years. We hope that ECE3084 reinforces concepts learned in ECE2026; in some cases, students manage to understand the "second time around" concepts that they didn't quite catch the "first time around."

Another example of spreading material through multiple courses is our coverage of circuits. Although we emphasize that signal and system theory is widely applicable, this text is unapologetically an electrical engineering text, so electric circuits play a central role, instead of being relegated to a few example problems. During the ECE2025 era, ECE2040 included the analysis of circuits using Laplace transforms. However, that portion of ECE2040 sometimes felt rushed, with Laplace transforms described at a somewhat superficial level. As part of the new curriculum, circuit containing capacitors and inductors with initial conditions are addressed using particular and homogenous solution methods,[2] and in ECE3084, they are revisited using Laplace-domain circuit equivalents in Chapter 14.

0.4 For *All* Electrical Engineers

"Signals and Systems" courses are usually taught by – and textbooks usually written by – faculty working in the areas of communications, control, and/or signal processing. However, such courses are usually required of all electrical engineering majors, even those not interested in those particular areas. At Georgia Tech, the staffing of ECE2026 is coordinated by the Digital Signal Processing Technical Interest Group (TIG), and ECE3084 is coordinated by the Systems and Controls TIG. This is an incidental administrative artifice; ECE2026 and ECE3084 represent parts one and two of a unified exposition on fundamental mathematical techniques that apply to nearly every aspect of electrical engineering, including optics, electromagnetics, and biophysics.

The "all" in the title also refers to the philosophy that such a required course should serve students with a wide variety of abilities and different strengths and weaknesses. Although a heavy dose of mathematics is unavoidable in such a course, we emphasize the development of intuition over the details of manipulating equations. We also unrepentantly employ phrases along the lines of "for most functions of practical interest in engineering applications" in preference to getting bogged down in discussions of Lebesque measurability and Hilbert space theory that are best left for graduate courses. That said, we do not shy away from addressing complicated theoretical issues when avoiding them could lead to misunderstandings and incorrect conclusions, particularly when it comes to dealing with unilateral Laplace transforms at the origin and impulse sources in electric circuits.

[2] This is the technique used in the MIT course 6.002: Circuits and Electronics.

0.5 Tough Choices

Should Fourier transforms be introduced before Laplace transforms, or Laplace transforms be introduced before Fourier transforms? Some authors consider the Fourier transform to be a special case of the Laplace transform. We prefer to present Fourier transforms first in ECE3084; in addition to better echoing the structure of ECE2026, we like to motivate the Laplace transform in terms of overcoming some limitations of the Fourier transform. Also, in many engineering applications, the unilateral Laplace transform is often sufficient; introducing the Fourier transform as a special case of the Laplace transform requires detailed discussion of the bilateral Laplace transform along with its "region of convergence" issues, which is time that we believe is better spent on other topics.

Many engineers (not just the electrical sort) experience the Laplace transform in the context of systems described by linear differential equations with constant coefficients, so the transforms are ratio of polynomials. Textbooks that begin with Laplace transforms might give the impression that such "lumped' systems are the only ones that are important. However, there are many "systems," particularly applications in *spatial domains*, such as optical point-spread functions and antenna beam patterns, that cannot be described by ratios-of-polynomials in the Laplace domain. Topics in optics and electromagnetics may be better suited to the Fourier domain, so we introduce it first partly to as a hat tip to these areas that are often neglected in junior-level "Signals and System" courses.

We also spent some time contemplating whether or not to cover random processes. This text is intended to cover a single junior-level semester-long course, and the schedule already feels fairly tight. Ultimately, proper coverage of random processes would necessitate dropping some other material, such as the chapter on PID control, which we prefer to keep as an example of feedback, and because nearly every electrical engineer will encounter PID control at some point, regardless of whether they've taken an elective class on control systems. At Georgia Tech, random processes are covered in ECE4260: Random Signals and Applications, a senior-level elective. The omission of random processes does constrain the way we cover cross-correlation and matched filtering, since we cannot present the matched filter from the standpoint of maximizing signal-to-noise ratios. Matched filtering is not part of most junior signals and systems courses, but like PID control, it arises in so many applications that we chose to cover it as an application of convolution.

Part I: Building Blocks

Chapter 1

What are Signals?

Let us begin by considering what we mean by the term “signals,” in the sense of a course with a name like “signals and systems.”

Mathematically, we abstract signals as functions that map a *domain* to a *range.* In particular, we are interested in domains that have some sort of ordered structure. Courses with titles like “signals and systems” typically focus on situations where the domain represents *time.* But can also have spatial-domain signals, in multiple dimensions, such as two-dimensional images and three-dimensional MRI scans. We may also have “space-time” signals; you can can think of a movie as a 3-D signal, where there are two spatial dimensions and one time dimension. Electromagnetic or acoustic waves propagating through space are often parameterized as four-dimensional signals with three spatial dimensions and one time dimension.

Our mathematical abstractions will let us apply the same tools in both time and space. By force of habit, we will usually use the term “continuous time,” and use a domain variable t, even when we are discussing ideas that would apply to “continuous space” or any other continuous domain.

Conventional mathematical notation can sometimes be a bit ambiguous without context. For instance, $x(t)$ might mean a function evaluated at a particular value of t, or we might write $x(t)$ to represent the entire function, in which case one might write x or $x(\cdot)$ instead.

1.1 Convenient continuous-time signals

Here, we will consider some continuous-time versions of *signals* that you previously encountered in your studies of discrete-time signals. We will discuss continuous time *systems* later.

1.1.1 Unit step functions

Recall the “unit step” function $u[n]$ (Fig. 1.1, left), defined as

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases} \tag{1.1}$$

In continuous-time, we have an obvious equivalent (Fig. 1.1, right):

$$u(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{1.2}$$

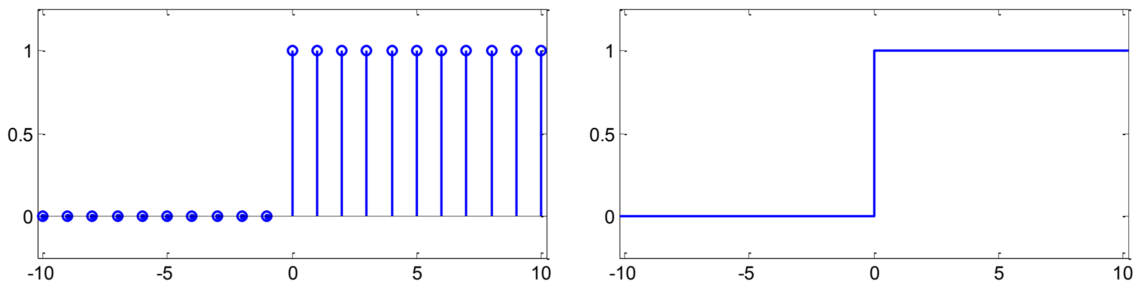


Figure 1.1: Discrete-time unit step (left) and continuous-time unit step (right).

The definition of the unit step at zero could be subject to debate. Here, we prefer not to say what $u(0)$ – basically the ”vertical line” is in the right panel of Fig. 1.1 – at all. In general, we will go to great pains to avoid situations where the exact value of $u(0)$ matters. Engineering textbook authors who like to precisely define $u(0)$ seem to most commonly pick $u(0) = 1$, since that is the most obvious analogy with the discrete-time version in (1.1), but one might also define it with $u(0)$. Another common choice, justified by some advanced but somewhat fussy mathematics, is to "split the difference" and let $u(0) = 1/2$. Again, we will prefer to just not talk about what $u(t)$ is at exactly 0 unless forced to. Any conclusions drawn from analyses in which the value of $u(t)$ at 0 is important should be viewed with a skeptical eye, such as in Section .

The use of the letter u, while a convenient shorthand for "unit," can lead to confusion because some authors, particularly ones working in the area of control theory, prefer to use $u(t)$ to represent generic input signals. Some authors prefer to use $q(t)$, $\mathbf{1}(t)$ or $H(t)$ to present the unit step. The latter comes from the continuous-time unit step sometimes being called the Heaviside step function, which is named after the self-taught English electrical engineer, mathematician, and physicist Oliver Heaviside.

Unit step functions are helpful for representing "one-sided" functions that "turn on" at some time. For instance, $x(t) = \cos(2\pi ft)u(t)$ and $x(t) = \sin(2\pi ft)u(t)$ are sinusoids that "turn on" at $t = 0$. The latter is continuous; the former is not.

Recall that we can time-shift functions; for instance, the expression $x(t-3)$ shifts a function $x(t)$ three"time units" to the right (these time units could be seconds or nanoseconds or centuries; there's usually some assumed time unit that we don't explicitly notate in the equations) and x(t+3) shifts a function three "time units" to the left.

Unit steps are convenient for writing "boxcar" functions, like $x(t) = u(t+2) - u(t-4)$, which is one between $t = -2$ and $t = 4$ and zero outside of that. (Since $x(t)$ is discontinuous at $t = -2$ and $t = 4$, we would generally try to avoid explicitly defining $x(-2)$ and $x(4)$; any analysis relying on having exact values of $x(-2)$ and $x(4)$ must be viewed with suspicion.) Some caution is needed; when writing an equivalent boxcar in discrete time, if we want to include the $n = 4$ point, we need to write $u[n+2] - u[n-5]$, so there is a slight difference between the analogous functions in discrete and continuous domains in terms of how the formulas looks.

Boxcars let us easily define finite-length signals by multiplying them by some other signal. For example, the function $y(t) = \exp(t)[u(t) - u(t-1)]$ "chops" out a piece of an exponential in the interval $0 < t < 1$ and is zero for $t < 0$ and $t > 1$. (At this point, we probably do not need to point out that we avoided saying what the values at exactly $t = 0$ and $t = 1$ are).

1.1.2 Delta "functions"

Recall the "Kronecker" delta function from your studies of discrete-time signals:

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \tag{1.3}$$

The Kronecker delta is an "impulse" in discrete-time. The continuous-time version of an impulse is called a "Dirac" delta function. A hand-waving and almost criminally misleading definition might look something like:

$$\delta(t) = \begin{cases} \text{infinity???} & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases} \tag{1.4}$$

Keep the question marks after the word "infinity" in mind; staring into the center of the Dirac delta function may induce madness. The trouble with the hand-waving definition is that the Dirac delta is *not an ordinary function* like what you might have seen in a freshman calculus class, and any function it appears in is not an ordinary function either. The Dirac delta is something called a *generalized function*, and generalized functions are properly defined in terms of how they *react to being integrated*.[1] If you integrate "over" a Dirac delta, you get 1:

$$\text{For } t_1 < t_2\text{: } \int_{t_1}^{t_2} \delta(t)dt = \begin{cases} 1 & \text{if } t_1 < 0 < t_2 \text{, i.e. } 0 \in (t_1, t_2), \\ 0 & \text{if } t_1 > 0 \text{ or } t_2 < 0 \text{, i.e., } 0 \notin [t_1, t_2]. \end{cases} \tag{1.5}$$

This is jarring because changing an ordinary function at a *single point* does not change the value of its integral; our Freshman calculus interpretation of the heuristic definition (1.4) would suggest that the integral of $\delta(t)$ over the whole real line should be zero. A tremendous amount of arcane graduate-level mathematical machinery has to happen under the hood for any of this to make sense. We will let the full-time mathematicians worry about such details, and stick with a distilled version of the theory that is both useful to and digestible by mere mortal engineers.

Notice that we have avoided precisely defining the "edge" cases where $t_1 = t_0$ or $t_2 = t_0$. This is related to our reluctance to precisely define $u(t)$ at $t = 0$.

Some more rigorous definitions of the Dirac delta function[2] involve a limiting sequence of functions that have unit area, and that get narrower and spikier, such as:

$$\delta_\Delta = \frac{u(t + \Delta/2) - u(t - \Delta/2)}{\Delta}. \tag{1.6}$$

We might then quasi-handwavily write $\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$.

You can shift the Dirac delta to be wherever you want, just as can be done with any other signal. For example, $\delta(t-5)$ shifts the Dirac delta five units to the right, and $\delta(t - t_0)$ is shifted t_0 units, which is a right-shift for $t_0 > 0$ and left shift for $t_0 < 0$.

When using a shifted Dirac delta, a simplification that often comes in handy, sometimes called the *sampling property*, is to rewrite $x(t)\delta(t - t_0)$ as $x(t_0)\delta(t - t_0)$; assuming $x(t)$ is continuous at t_0, this works since once you multiply $x(t)$ by $\delta(t - t_0)$, it does not matter what $x(t)$ was for any point besides $t = t_0$. For

[1] At this point, we must admit that our characterization of signals earlier as functions that map some domain to some range should be viewed as incomplete, now that we have introduced these bizarre "generalized functions."

[2] To be precise, one should call it the "Dirac delta *generalized* function," but nobody actually does that. Just make sure that every time you see it you chant to yourself, "it's not an ordinary function, and if I treat it like one without paying close attention, I will get myself into trouble."

instance, we could simplify $t^3\delta(t+2)$ as $(-2)^3\delta(t+2) = -8\delta(t+2)$. (A common mistake when doing this simplification is to forget to include the δ part – $\delta(t+2)$ in this example – in the simplified version.)

Using that idea and the Dirac integral definition yields the *sifting property*, which evaluates a function at a particular point by integration:

$$\int_{t_1}^{t_2} x(t)\delta(t-t_0)dt = \begin{cases} x(t_0) & \text{if } t_0 \in (t_1, t_2), \\ 0 & \text{if } t_0 \notin [t_1, t_2], \end{cases} \tag{1.7}$$

where we once again hedge about cases where $t_1 = t_0$ or $t_2 = t_0$.

One often sees a one-line instantiation of this with $t_1 = -\infty$ and $t_2 = \infty$:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \tag{1.8}$$

1.1.3 Calculus with Dirac deltas and unit steps

Notice that if we integrate over a Dirac delta from $-\infty$ to t, we get a unit step:

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t). \tag{1.9}$$

The unit step $u(t)$ is discontinuous at $t = 0$, so by the ordinary rules of freshman calculus, you cannot take its derivative at $t = 0$. But in the quirky calculus of generalized functions, we can write

$$\frac{du(t)}{dt} = \delta(t), \tag{1.10}$$

and it will make sense in terms of the fundamental theorem of calculus if we integrate both sides.

Example: Here, we shall differentiate a decaying exponential that turns on at $t = 4$, namely $x(t) = e^{-3(t-4)}u(t-4)$.

Using the rules for the derivative of products,

$$\frac{dx(t)}{dt} = e^{-3(t-4)}\delta(t-4) - 3e^{-3(t-4)}u(t-4). \tag{1.11}$$

Using the sampling property, we can simpify the first term by plugging 4 in for t in $e^{-3(t-4)}$, yielding

$$\frac{dx(t)}{dt} = \delta(t-4) - 3e^{-3(t-4)}u(t-4). \tag{1.12}$$

Forgetting to include the $\delta(t-4)$ term is a common error.

1.2 Shifting, flipping and scaling continuous-time signals in time

There are several common manipulations of continuous time signals that merit review. Time shifting, which was mentioned previously, is perhaps the most common, where $x(t-t_0)$ is a shifted version of $x(t)$. Don't forget that $t_0 > 0$ is a shift to the right (negative sign), whereas $t_0 < 0$ is a shift to the left (positive sign). "Flipping" a signal in time simply refers to mirroring it about $t = 0$, where $x(-t)$ is the flipped version of $x(t)$. Time scaling a signal refers to either stretching it or compressing it in time, where $x(at)$ is a scaled version of $x(t)$. If $|a| > 1$, the signal is actually compressed instead of stretched, which can seem counter-intuitive. Note that $a = -1$ corresponds to flipping a signal in time.

While each of these individual manipulations is generally straightforward, it can be confusing to apply multiple ones such as shifting and flipping, or shifting and scaling. Looking at specific examples is useful, several of which are shown in the following figure. Note that the star symbol on each plot corresponds to the point of the original signal that was at $t = 0$.

In general, it is easier to think of a signal as first being scaled and then being shifted; that is, express a scaled and shifted signal as $x(a(t - t_0))$ rather than $x(at - b)$. Then it is clear that the signal is first scaled by a, followed by shifting the scaled signal $x(at)$ by t_0.

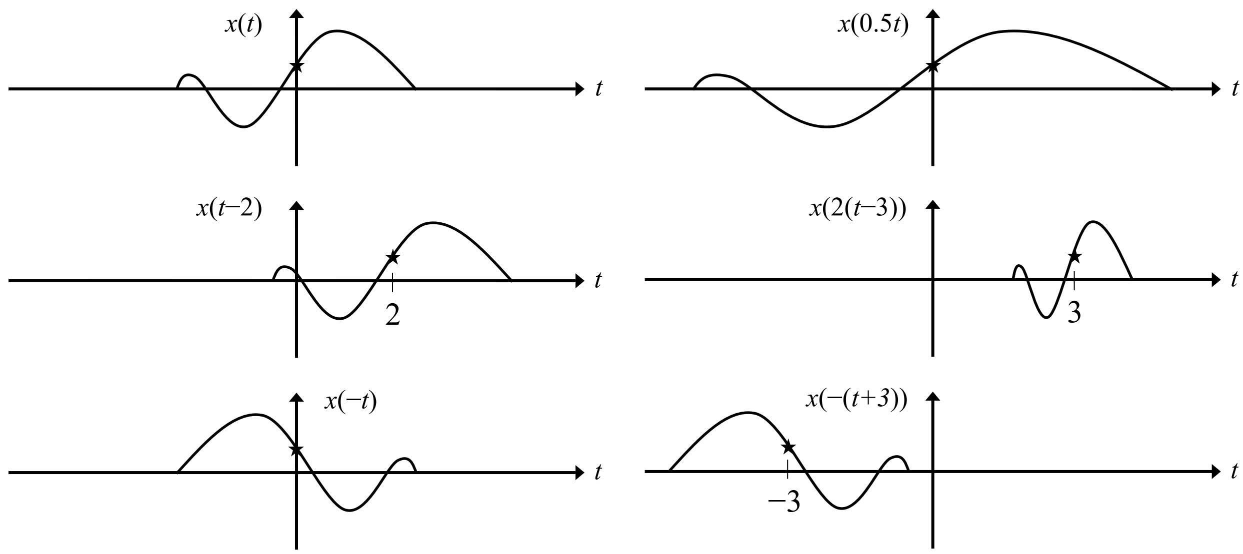


Figure 1.2: Examples of time shifting, scaling, and flipping.

1.3 Under the hood: what professors don't want to talk about

We spent a good portion of this chapter avoiding talking about what happens *exactly* at the discontinuity of $u(t)$, and what the integral of a Dirac delta is if one of the limits of the integral is located *exactly* on the Dirac delta. This section will informally explore a few possibilities, mostly to establish that it is difficult to develop a satisfying, consistent "answer." Although not many textbooks venture into this territory, particularly inquisitive students always seem to ask about it.

Let's suppose that we decided to set $u(0) = 1$; many textbooks use this choice for $u(t)$. To avoid ambiguity, we will use the subscript geq, for greater than or equal to, to specify this choice of step function:

$$u_{\text{geq}}(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{1.13}$$

We would like our unit step function to be able to act as the antiderivative, aka, the indefinite integral, of the Dirac delta. Trudging along symbolically, we might write

$$\int_{-\infty}^{0} \delta(t)dt = u_{\text{geq}}(0) - u_{\text{geq}}(-\infty) = 1 - 0 = 1 \tag{1.14}$$

and

$$\int_{0}^{\infty} \delta(t)dt = u_{\text{geq}}(\infty) - u_{\text{geq}}(0) = 1 - 1 = 0. \tag{1.15}$$

This is profoundly unsettling. Our attempt to "derive" the dirac Delta function as a limiting function of rectangles centered around the origin gives no hints as to why we should expect a delta function integrated between $-\infty$ and 0 to yield a different result than when integrated between 0 and ∞. We would need to revise (1.5) to look like

$$\text{For } t_1 < t_2\text{: } \int_{t_1}^{t_2} \delta(t)dt = \begin{cases} 1 & \text{if } t_1 < 0 \leq t_2 \text{, i.e. } 0 \in (t_1, t_2], \\ 0 & \text{if } t_1 \geq 0 \text{ or } t_2 < 0 \text{, i.e., } 0 \notin (t_1, t_2], \end{cases} \tag{1.16}$$

and it seems quite odd to have the interval in the if-then statement be open on the left and closed on the right. Choosing a $u_{\text{leq}}(t)$ with $u_{\text{leq}}(0) = 0$ would change the interval in (1.16) to read $[t_1, t_2)$.

Let's suppose that we "split the difference" and decide to set $u(0) = 1/2$. To avoid ambiguity, we will use the subscript "half" and write

$$u_{\text{half}}(t) = \begin{cases} 1 & \text{for } t > 0, \\ 1/2 & \text{for } t = 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{1.17}$$

This at least gives us a consistent answer to the question about what might happen when the location of the Dirac delta function lands exactly at one of the limits of the integral:

$$\int_{-\infty}^{0} \delta(t)dt = u_{\text{half}}(0) - u_{\text{half}}(-\infty) = 1/2 - 0 = 1/2 \tag{1.18}$$

and

$$\int_{0}^{\infty} \delta(t)dt = u_{\text{half}}(\infty) - u_{\text{half}}(0) = 1 - 1/2 = 1/2. \tag{1.19}$$

This somewhat fits our derivation of a Dirac delta as a limit of rectangles, since the set of limits $-\infty$ to 0 and 0 to ∞ each cover half of a particular rectangle. But it seems awfully nitpicky to keep track of all of these one-halves, and doing so is rarely helpful in practice. There are a few textbooks that use this nitpicky definition.[3]

Hiding in the background is the question of whether the actual limits of the integral are included in the range of values integrated over. When integrating "ordinary" functions, it doesn't matter. But when considering dirac Delta functions – which, once again, are not really functions in the usual sense – it can matter. We could hope to avoid this situation, and only deal with issues that arise in typical engineering practice. Unfortunately in our case, this issue arises rather dramatically in the context of unilateral Laplace transforms, which play a vital role in the second half of this book. Many textbooks treat this issue in a somewhat haphazard and inconsistent way, which can lead to great confusion.

[3] This business with the "1/2" at the transition might return to haunt us when we discuss Fourier series and Fourier transforms, depending on the desired level of mathematical rigor.

Our "out" will be to augment the usual notation of limits in an integral to make it clear whether or not the edge is included. For instance, we can write

$$\int_{0^-}^{\infty} \delta(t)dt = 1, \tag{1.20}$$

where the superscript minus sign indicates that the integral extends to "just past" the lower limit and hence include the Dirac delta. One might define, for $a < b$,

$$\int_{a^-}^{b} x(t)dt \equiv \lim_{\alpha \to a^-} \int_{\alpha}^{b} x(t)dt, \tag{1.21}$$

where the superscript minus sign below the "lim" indicates that α approaches a from below.

Chapter 2

What are Systems?

In Chapter 1, we described *signals* as functions mapping from some domain, such as time or space, to some range, such as air pressure, voltage, or light intensity.

We will abstract *systems* as mappings from one set of signals to another set of signals. One could think of systems as functions, where the domain and range of these functions (systems) are themselves functions (signals). Such mappings are often called "operators."

We will generally refer to input signal as $x(t)$ and output signals as $y(t)$.[1]

Consider the ideal continuous-to-discrete sampler defined by $y[n] = x(T_s n)$, where T_s is the sample period. The sampler maps continuous signals to discrete-time signals:

$$(\Re \to \Re) \to (\mathbb{Z} \to \Re)$$

In the above representation, the left side denotes that the domain and range of the input are both the set of real numbers, whereas on the right the domain is the set of integers (the time samples) and the range is the set of real numbers (the values that can be assumed at each time point). For an actual analog-to-digital converter, both the domain and range would be quantized, but here we assume that all digital signals are only quantized in time (i.e., the domain).

The field of "digital signal processing" mostly focuses on systems that map discrete-time signals to discrete-time signals:

$$(\mathbb{Z} \to \Re) \to (\mathbb{Z} \to \Re)$$

This text primarily focuses on systems that map continuous-time signals to continuous-time signals (although discrete time will pop its head in the door from time to time):

$$(\Re \to \Re) \to (\Re \to \Re)$$

Complex-valued signals are sometimes useful too, particularly in communications and radar 8.4, but we will mostly focus on real-valued signals. In particular, we will focus on single input–single output (SISO)

[1] Some authors use other conventions; for instance, control systems engineers often use $u(t)$ to represent generic inputs; we avoid that convention since it conflicts with our use of $u(t)$ to represent a unit step function. All of the concepts we discuss in the context of functions of time could also be applied to functions of space. Spatial coordinates are usually labeled x, y, and z; the use of x to denote a function conflicts with that, so authors focusing on spatial domains, such as image processing, often use f to represent functions of spatial coordinates, such in $f(x, y)$. For additional excitement, some authors use different conventions on different pages of the same book.

systems, for which there is one input and one output, as opposed to the more general multiple input–multiple output (MIMO) systems, for which there can be multiple inputs and outputs.

Caveat: Our current conceptualization of systems – which is what often comes to mind for engineers who work in fields broadly defined as "signal and information processing" and "telecommunications" – is incomplete. It leaves out the notion of "initial conditions" – i.e., that the system may be in some non-restive state before we start poking at it with an input signal. In fact, one can put a great deal of effort into studying the behavior of systems with *no input at all* once initial conditions are "put into the equation" (both figuratively and literally). Such issues are of particular importance in control system engineering. We will revisit this issue later in Chapter 10.

2.1 System properties

The concepts of linearity, time-invariance, and causality defined in the context of discrete-time systems transfer naturally to continuous-time systems. The idea of time-invariance extends to other kinds of domains; for instance, if the domain is spatial instead of temporal, one could use the the term "space-invariance." By force of habit, we will often use the term time-invariance when discussing properties that could more generally be called "shift invariance."

The concept of causality is most natural when the domain has some inherent "direction," as in the case of time.

It is good to develop intuition about whether a system is linear, time-invariant, and/or causal. This section will present some "back of the brain" tricks that will serve you well much of the time.

Another important property of systems is *stability*, but we will hold off on that for now. We will address stability in later chapters after we have built more mathematical scaffolding.

2.1.1 Linearity

A system is said to be linear if linear combinations of inputs result in the same linear combinations of outputs. A system can readily be tested for linearity by considering scaled and summed inputs. Suppose that a SISO system with input $x_1(t)$ results in the corresponding output $y_1(t)$, and input $x_2(t)$ results in the output $y_2(t)$. The system is linear if the input $x(t) = A_1x_1(t) + A_2x_2(t)$ results in the output $y(t) = A_1y_1(t) + A_2y_2(t)$ for all functions $x_1(t)$ and $x_2(t)$ and all scalars A_1 and A_2. It is often convenient to show that a system is nonlinear by counterexample; that is, by showing at least one case for which the above is not true.

Intuitively, "messing" with the x's often "breaks" linearity. Consider the following examples:

- Consider the system $y(t) = [x(t)]^p = x^p(t)$ (this is a common notation for powers of functions). This system is only linear if $p = 1$.

- The system $y(t) = 3x(t) + 5$ is not linear in the sense of a "linear system" because of the second term. This is a little tricky, since the function $y = 3x + 5$ defines a straight line, and is "linear" in that sense. Confusion arises from the term "linear" having different definitions in different contexts. The transformation $y(t) = 3x(t) + 5$ can be called "affine." For our purposes, the $+5$ would not present too much of a problem; one could just leave it out, study the system using all the linear system techniques we will develop, and then add the 5 back in later.

2.1.2 Time-invariance

A system is said to be time-invariant if a time-shifted input results in the output being time shifted by the same amount. A system can readily be tested for time-invariance by considering an input and its time-shifted version. Suppose that a SISO system with input $x_0(t)$ results in the corresponding output $y_0(t)$. The system is time-invariant if the input $x_1(t) = x_0(t - t_0)$ results in the output $y_1(t) = y_0(t - t_0)$ for all inputs $x_0(t)$ and all time shifts t_0. It is often convenient to show that a system is not time-invariant by counterexample; that is, by showing at least one case for which the above is not true.

Intuitively, time-invariance is often "broken" by one of two things:

1. Seeing a t outside of the argument of $x(\cdot)$. For instance, the systems $y(t) = tx(t)$ and $y(t) = x(t) + t$ are not time-invariant.

2. "Messing" with the t in the argument $x(\cdot)$. For instance:

   - The system $y(t) = x(-t)$ is linear, but not time-invariant; shifting the input into the future shifts the output into the past and vice-versa.
   - The time-compressing system $y(t) = x(3t)$ is not time-invariant.
   - The time-expanding system $y(t) = x(t/3)$ is not time-invariant.
   - The system $y(t) = x(t^2)$ is not time invariant.

   The only "safe" thing to do with the arguments of x, in terms of maintaining time-invariance, is to add or subtract constants from a well-behaved, not-messed-with t variable. For instance, $y(t) = x(t + 3) + x(t - 4.8)$ is time-invariant.

Not having a t variable in the argument of x at all is also problematic in terms of time-invariance. For instance, the system $y(t) = x(3)$, is not time-invariant.

2.1.3 Causality

A system is causal if, for all t, the output $y(t)$ is not a function of $x(\tau)$ for any $\tau > t$. In other words, causal systems cannot look into the future. They might or might not look at the present, and they might or might not look into the past.

For time-invariant systems, causality, or the lack thereof, is usually pretty easy to determine – one can look at the arguments of the x instances, and check to see whether there are positive constants added to t in those arguments. Such instances look into the future and break causality. Sometimes this kind of reasoning helps in considering the causality of non-time-invariant systems as well. Consider these examples:

1. The system $y(t) = t\sqrt{x(t - 4)}$ is not linear or time-invariant, but it is causal; it obviously does not look into the future.

2. The system $y(t) = x(t + 4)$ is linear and time invariant, but it is not causal because it does look into the future.

One can cook up unusual cases in which a bit more thinking is required to evaluate causality, although these are admittedly more mathematical curiosities that practical systems. Consider the following:

1. The discrete-time system $y[n] = x[-n^2]$. This is clearly linear but not time-invariant. To gain insight into causality, or lack thereof, try a few values of n: $y[3] = x[-9]$
$y[2] = x[-4]$
$y[1] = x[-1]$
$y[0] = x[0]$
$y[-1] = x[-1]$
$y[-2] = x[-4]$
$y[-3] = x[-9]$

   The system is always looking into the present or the past. So, although having n^2 by itself without the minus sign would force the system have to look into the future, having the minus sign in front of the n^2 facilitates causality.

2. Now consider the system $y[n] = x[-n^3]$. We see that $y[3] = x[-27]$, but $y[-3] = x[-(-27)] = x[27]$, which looks into the future. Hence, this system is not causal.

3. What about the continuous-time equivalents of the above? The logic we used on $y[n] = x[-n^3]$ quickly shows that $y(t) = x(-t^3)$ is noncausal as well: $y(-3) = x(27)$.

4. Our final example in this sequence is tricky. Consider $y(t) = x(-t^2)$. We might be tempted to believe that we could just change the brackets in our above "trial runs" for the discrete-time example $y[n] = x[-n^2]$ into parentheses and conclude that $y(t) = x(-t^2)$ was causal – but we would be wrong! The region $t \in (0, 1)$ crashes the causality party – it's a weird zone[2] we did not encounter in the discrete-time case, in which t^2 is smaller than t. For instance, $y(-1/2) = x(-1/4)$, which looks into the future.

2.1.4 Examples of systems and their properties

1. **Systems defined by derivatives and integrals of the input** inherit their linearity and time-invariance from the properties of taking derivatives and integrals of sums from your calculus class. But you can also see the linearity "directly" by contemplating one of the definitions of a derivative:

$$\frac{dx(t)}{dt} = \lim_{\epsilon \to 0^+} \frac{x(t) - x(t-\epsilon)}{\epsilon},$$

   where the + superscript indicates that ϵ approaches 0 "from above." There is nothing "messing with" the xs, there are no ts outside of the arguments of x, and nothing is "messing with" the ts inside the arguments of x.

   You can think of applying the same ideas to the terms in the Riemann sum definition of the integral. (Professional mathematicians would want to put a dozen caveats here; in particular, expressions with Dirac deltas do strange things that require special treatment, but we will not worry about most of these details.)

   So, we have convinced ourselves that

$$y(t) = \frac{dx(t)}{dt}$$

[2] This issue was pointed out to Aaron Lanterman by an ECE2025 student a few years back.

is linear, and by the "left derivative" definition from Freshman calculus definition shown above, it is also causal. However, calculus books also define a "right derivative" as

$$\frac{dx(t)}{dt} = \lim_{\epsilon \to 0^+} \frac{x(t+\epsilon) - x(t)}{\epsilon}$$

which looks like it would not be causal, because the $x(t+\epsilon)$ term looks into the future.

As will usually be the case when we run into sticky technical details, we will apply a bit of handwaving. If the left and right derivatives are the same, the function is differentiable, and we might as well pick the left one when talking about causality. We have to be a little careful since we have introduced the Dirac delta as a derivative of a unit step function. That relationship is more amenable to the "left derivative" definition. So we will consider derivatives (and higher-order derivatives) to be causal.

2. The **"slicer"** $y(t) = x(t)x^\star(t - T_s)$, where the $\star$ represents complex conjugation and $T_s > 0$ is a sample period, is sometimes used in modems. It is causal (since $T_s \geq 0$) and time-invariant, but it is nonlinear because of the multiplication.

3. The transmitter of many communication schemes falling under the umbrella of **amplitude modulation** can be modeled as $y(t) = [A + x(t)]\cos(\omega_c t)$. The $\cos(\omega_c)$ is called the *carrier* and ω_c is the *carrier frequency*. It is not linear in general, but it is in the special case of $A = 0$ (which, incidentally, is referred to as "suppressed carrier"). It's not time-invariant *in general* because of the t appearing outside the x argument, but it can be made time-invariant in the trivializing special case $\omega = 0$.

   Real AM broadcast transmitters use an A that is greater than the maximum amplitude of $x(t)$. This is theoretically inefficient from an information-theoretic viewpoint, but it allows the design of AM receivers to be simple and inexpensive. We will look at amplitude modulation in Section 8.3 as an application of Fourier transform properties.

4. The nihilistic system $y(t) = 0$ is trivially linear, time-invariant, and causal. It doesn't depend on the future, but it doesn't depend on the present or the past either. The slightly less nihilistic but still not terribly interesting system $y(t) = 3$ would be trivially time-invariant and causal, but it is not linear.

2.2 Concluding thoughts

2.2.1 Linearity and time-invariance as approximations

Few physical systems are truly linear. In designing "linear amplifier" circuits, transistors are operated within a region in which its voltage and current relationships may be approximated as linear, and circuits are described as a linear system (the "a.c. small signal" model) with a systematic bias (the "d.c. operating point").

Few physical systems are truly time-invariant. The properties of resistors and transistors–particularly the latter–change with temperature, and electronic equipment generally heats up gradually after it has been turned on. Circuit designers must sometimes go to great length to compensate for these effects. We often resort to saying that a system is time-invariant over the time scales of interest while realizing that behavior may change over long time scales.

When resistors are supplied so much power that they start to smell bad, they can quickly become nonlinear and then highly time-variant as they explode.

2.2.2 Contemplations on causality

The term anticausal applies to systems that do not look into the past. They might or might not look at the present, and they might or might not look into the future. The texts focuses on causality since it is usually the more relevant concept.

Obviously, "real-time systems" – for instance, the effects processors a guitarist is using on stage – have to be causal, but it is also reasonable to talk about noncausal systems in cases where there is a set of prerecorded data that can be processed "offline" (like applying some effects processing to a track that has been recorded in Garageband). Nowadays, that will usually be some kind of digital recording, but one you could imagine an old-school analog recording on magnetic tape with a playback head set in to look into the "future."

This chapter has considered time-domain examples. Everything we have discussed also applies to one-dimensional spatial-domain systems, and although "causality" and "anticausality" can still be defined if you swap in a spatial variable for the time variable (for instance, one could define the "future" is as being to the right and the "past" as being to the left), "causality" is both a less important property and a rarer property of spatial systems. Additionally, once we start considering two-dimensional and three-dimensional domains, we lose a sense of directionality, so people who study image processing do not use the term "causal" very often.

2.2.3 How these properties play out in practice in a typical "signals and systems" course

Professors like to make students suffer through a few homework and/or quiz problems where they have to circle "yes" or "no" as to whether a system is linear, time-invariance, and/or causal. After that, they usually spend the rest of the semester lecturing almost exclusively about linear, time-invariant (LTI) systems. This is because (a) LTI systems are amenable to a rich set of convenient mathematical techniques; (b) enough real, practical systems can be modeled as LTI systems that we can take advantage of (a); and (c) nonlinear systems can be incredibly difficult to analyze.

Part II: Time-Frequency Duality

Chapter 3

Why are LTI Systems so Important?

This chapter explores the question: why are LTI systems so interesting?

This chapter is about the "big" ideas. It is basically a 30,000 feet view of the the first half of this text (which sets the stage for the second).

This chapter will not contain many concrete examples. Our goal here is to paint a self-contained synopsis of the some of the main themes of this subject, and to do that, we will need to hold a big brush.

3.1 Review of convolution for discrete-time signals

In 2026, we looked at the idea of an impulse response to a system, $h[n]$, which is the special output you get when you input a delta function $x[n] = \delta[n]$.

The beauty of time-invariance is that if we know the response of the system to a delta at the origin, we know the response of a δ at any point. Putting $x[n] = \delta[n - n_0]$ into a time-invariant system gives you an output of $y[n] = h[n - n_0]$.

The beauty of linearity is that if we know the response of a system to a "unit weighted" delta, i.e., a weight of 1, you know the response of the output to deltas of any weight. Putting $x[n] = \alpha\delta[n]$ into a a linear system gives you an output of $y[n] = \alpha h[n]$.

The last two beautiful observations tell us that if we put $x[n] = \alpha\delta[n - n_0]$ into an LTI system, we get an output of $y[n] = \alpha h[n - n_0]$ out.

We invoked one more property of linearity – superposition – to note that if we write the input as a sum of weighted, shifted delta functions:

$$x[n] = \sum_k x[k]\delta[n - k], \tag{3.1}$$

we can express the output as a sum of weighted, shifted impulse responses:

$$y[n] = \sum_k x[k]h[n - k]. \tag{3.2}$$

We gave this the somewhat convoluted name of *convolution*, notated like $y[n] = x[n] * h[n]$. That common notation is a bit misleading, since it makes you think that you evaluate $f[\cdot]$ at n and evaluate $h[\cdot]$ at n, find two scalars, and then do something with those scalars. But convolution is an operation on entire functions; a clearer notation might be $y[n] = (f * h)[n]$.

We saw that convolution was commutative (we can flip the order of the arguments to a convolution) and associative (we can rearrange parentheses in a set of convolutions). This let us rearrange the order of LTI systems.

3.2 Convolution for continuous-time signals

Once we accept the delightfulness of those devious Dirac delta functions, everything we have just said applies to continuous-time systems. We can imagine sacking the system with an infinitely quick, infinitely strong hammer called $\delta(t)$, seeing what pops out, and calling it $h(t)$.

Putting $x(t) = \delta(t - t_0)$ into a time-invariant system gives you $y(t) = h(t - t_0)$ out.

Putting $x(t) = \alpha\delta(t)$ into an linear system gives you $y(t) = \alpha h(t)$ out.

Putting $x(t) = \alpha\delta(t - t_0)$ into an linear system gives you $y(t) = \alpha h(t - t_0)$ out. (We build this slowly try to emphasize specifically where the linearity aspect and the time-invariance aspect come to play).

The next part might look a little weird. We (as engineers; mathematicians might want to state a few more conditions here) can write an input as a sum of weighted, shifted Dirac delta functions,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau, \tag{3.3}$$

and use superposition to write the output as a sum of weighted, shifted impulse responses,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau. \tag{3.4}$$

Do not be alarmed by the appearance of the integral signs. Remember that integrals are “just sums.” (We put “just sums” in air quotes, because under the surface, there are some mathematicians paddling like mad to make that work, especially when Dirac deltas are involved. Take a moment to appreciate their hard work). Summation signs let us sum over countably infinite sets, and integrals let us sum over uncountably infinite sets. Yes, there are different kinds of infinities. Countably infinite sets can be put in a one-to-one correspondence with the integers; for instance, the set of rational numbers is countable. The set of real numbers is not countable; it is a “bigger” kind of infinity.

We will use the same convolution notation for continuous domains as we used for discrete domains: $y(t) = x(t) * h(t)$ or $y(t) = (x * h)(t)$. Continuous-time convolution is commutative and associative as well, so we can rearrange the order of continuous-time LTI systems just as we could discrete-time LTI systems.

The mechanics of working out specific convolutions, particularly in continuous time, tend to be pretty messy and involve a lot of bookkeeping. We will save such unpleasantries for Chapter 4.

3.3 Review of frequency response of discrete-time systems

A particularly elegant aspect of LTI systems is the way they respond to sinusoids.

In 2026, we looked at how discrete-time LTI systems responded to sinusoids. A particularly convenient sinusoid is the *complex sinusoid* manifest by Euler’s formula:

$$\exp(j\hat{\omega}n) = \cos(\hat{\omega}n) + j\sin(\hat{\omega}n). \tag{3.5}$$

The complex[1] exponential is algebraically convenient, but when you see it, you should imagine wavy lines, just like you do when you see $\cos(\cdot)$ or $\sin(\cdot)$.

In 2026, we put $x[n] = \exp(j\hat{\omega}n)$ into a system; using convolution, found that the output was:

$$y[n] = \sum_k h[k]f[n-k] = \sum_k h[k]e^{j\hat{\omega}(n-k)} = \sum_k h[k]e^{j\hat{\omega}n}e^{-j\hat{\omega}k} = e^{j\hat{\omega}n}\underbrace{\sum_k h[k]e^{-j\hat{\omega}k}}_{\equiv H(e^{j\hat{\omega}})}. \tag{3.6}$$

We called $H(e^{j\hat{\omega}})$ the *frequency response*. In general, $H(e^{j\hat{\omega}})$ is complex-valued, and most naturally interpreted in polar form with a magnitude and a phase. If we feed a discrete-time LTI system a complex sinusoid, it chews it up and spits out the same complex sinusoid multiplied by a complex constant—essentially, the complex sinusoid's magnitude and phase may be changed, but the frequency stays the same.

The e^j in $H(\cdot)$ is a notational convention that provided a nice link to z-transforms, but we should note that some books just use $H(\hat{\omega})$, and some books leave out the $\hat{}$ for discrete-time frequencies (which can lead to much confusion), or use another notation like Ω.

By the scaling property of linearity (note that the scale factor can be complex), inputting a more general complex sinusoid, like $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, yields an output of

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n}H(e^{j\hat{\omega}}) = A|H(e^{j\hat{\omega}})|\exp(j[\phi + \angle\{H(e^{j\hat{\omega}})\}])e^{j\hat{\omega}n}. \tag{3.7}$$

In 2026, we also found the output arising from a real sinusoidal input via Euler's formula and linearity. An input of

$$x[n] = A\cos(\hat{\omega}n + \phi) = \frac{A}{2}e^{j\phi}e^{j\hat{\omega}n} + \frac{A}{2}e^{-j\phi}e^{-j\hat{\omega}n}. \tag{3.8}$$

produces the output

$$\begin{aligned} y[n] &= \frac{A}{2}e^{j\phi}e^{j\hat{\omega}n}H(e^{j\hat{\omega}}) + \frac{A}{2}e^{-j\phi}e^{-j\hat{\omega}n}H(e^{j(-\hat{\omega})}) && (3.9)\\ &= \frac{A}{2}e^{j\phi}e^{j\hat{\omega}n}H(e^{j\hat{\omega}}) + \frac{A}{2}e^{-j\phi}e^{-j\hat{\omega}n}H^*(e^{j\hat{\omega}}) && (3.10)\\ &= \frac{A}{2}|H(e^{j\hat{\omega}})|\exp(j[\phi + \angle\{H(e^{j\hat{\omega}})\}])e^{j\hat{\omega}n} + \frac{A}{2}|H(e^{j\hat{\omega}})|\exp(j[\phi + \angle\{H(e^{-j\hat{\omega}})\}])e^{-j\hat{\omega}n} && (3.11)\\ &= |H(e^{j\hat{\omega}})|\cos(\hat{\omega}n + \phi + \angle\{H(e^{j\hat{\omega}})\}). && (3.12) \end{aligned}$$

This was one of the "big ideas" from ECE2026: SINUSOID IN $\rightarrow$ SINUSOID OUT (for LTI systems). The magnitude of the frequency response, evaluated at the input frequency, multiplies the input amplitude, and the phase of the frequency response, evaluated at the input frequency, is added to the phase. The frequency stays the same.

That was just a review of ECE2026 material; now we make the transition to ECE3084.

3.4 Frequency response of continuous-time systems

You should be able to guess what comes next.

[1] Are complex numbers profound expressions of the nature of the universe, or just a convenient bookkeeping trick? You decide.

Let us put $x(t) = \exp(j\omega t)$ into a continuous-time LTI system. The resulting output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega t}e^{-j\omega\tau}d\tau = e^{j\omega t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}_{\equiv H(j\omega)}. \quad (3.13)$$

You should not be surprised to discover that $H(j\omega)$ is referred to as the *frequency response* of the system.

Including the j in the argument of $H(\cdot)$ is a notational convention. It helps differentiate continuous-time frequency responses form discrete-time frequency responses (assuming you are using the e^j in the argument of discrete-time frequency responses). Just as the e^j notation created an elegant link with z-transforms, we will see that the $j\omega$ creates makes an elegant link with the Laplace transforms introduced in Chapter 10.

Taking the real part of a complex signal is a LTI operation, and we can reorder LTI operations. So if we take the real part of that input and that output, we find that an input of

$$x(t) = A\cos(\omega t + \phi) \quad (3.14)$$

produces

$$y(t) = A|H(j\omega)|\cos(\omega t + \phi + \angle\{H(j\omega)\}). \quad (3.15)$$

The steps needed to get from (3.14) to (3.15) are analogous to those used to get from (3.8) to (3.12).

Different domain, same deal: we have SINUSOID IN → SINUSOID OUT (for LTI systems). If you remember nothing else from ECE3084, remember that!

3.5 Connection to Fourier transforms

The mathematical operations that we applied to find frequency responses can be applied to general signals, not just impulses responses:

$$X(e^{j\hat{\omega}}) = \sum_n x[n]e^{-j\hat{\omega}n}, \quad (3.16)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \quad (3.17)$$

We previously used k and τ as dummy variables to avoid being confused with other time variables; we do not need to worry about such confusion here, so it is acceptable to change them to n and t.

Equation 3.16 is called the discrete-time Fourier transform (DTFT); (3.17) is the continuous-time Fourier transform (CTFT). The latter is considered the mother of all Fourier transforms, so when someone just says "Fourier transform" without any additional qualifiers, they usually mean the continuous-time version.

You should never let anyone tell you about their transform without also asking for the inverse transform:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\omega, \quad (3.18)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \quad (3.19)$$

These inverse transforms look quite similar to the forward transforms; the main difference is the sign of the exponent. For now, you can take these on faith; in the next few chapters, we will show show you why the continuous-time pair works. The limits of the inverse DTFT integral arise from the periodicity of discrete-time frequencies—any interval of 2π will do. The DTFT pair is one of the main topics of ECE4270, the senior-level DSP class; you may also have seen it touched on in ECE2026. We mention it here for context; it will not be a player in ECE3084, which will focus on the continuous-time Fourier pair.

The $1/(2\pi)$ constants in the inverse transforms are awkward, but they are needed to get math to work out.

3.6 Finishing the picture

If you have been paying close attention, you will notice that although our discussion of frequency responses paralleled our discussion of impulse responses, we did not quite finish the frequency response discussion.

We talked about a scaled and shifted impulse giving us a scaled and shifted impulse response. We talked about a scaled and shifted (i.e. the phase) sinusoid giving us a scaled and shifted sinusoid.

We ended the discussion of impulse responses by talking about inputs that were *sums* of impulses—perhaps uncountably infinite sums, in the case of continuous-time systems. So to complete the story of frequency response, we need to talk about inputs that are sums of sinusoids. Take a look at (3.19), the inverse Fourier transform—that is exactly what it is! It is a "sum" (possibly uncountably infinite sum) of weighted sinusoids with different frequencies ω, where the weights are given by $X(j\omega)$. So by the superposition property, the output is

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega. \tag{3.20}$$

So, in the first few chapters of this book, you have already come across one of the most celebrated properties of Fourier transforms: convolution in the time domain corresponds to multiplication in the frequency domain. This is usually presented as a convenience: "convolution is in the time domain is hard, but it is much simpler to do the frequency domain, so this property can make your life easier." But the subtext is deeper than that; this property is just a manifestation of the fundamental concept of "SINUSOID IN $\rightarrow$ SINUSOID OUT" (for LTI systems).

We could make the same observations about the inverse DTFT, but we will leave that for specialized treatises on "digital signal processing."

3.7 A few observations

Before starting the next chapter, ponder the following points:

- Our discussion in this chapter assumed that the sums and integrals defining frequency responses and Fourier transform existed, i.e., they did not blow up to infinity or do something else antisocial like be entirely undefined. In the later chapters, we will look at Laplace transforms, which allow us to analyze signals for which the Fourier transform would be undefined; they can be viewed as a way of taming ornery functions. This is of particular value to control systems engineers, since much of control theory is concerned with taming ornery systems. (Similarly, the z-transform can be seen as a way of taming ornery discrete-time functions for which the DTFT would be undefined).

- There many different ways to represent functions as sums of some fundamental building blocks. Fourier representations use sinusoids, but there are also creatures called Chebyshev polynomials, Walsh and

Harr functions, wavelets, and so on. Sinusoids are particularly important because they are "eigenfunctions" of LTI systems (analogous with the eigenvectors you learn about in linear algebra; the frequency response is analogous to the eigenvalues), giving us their celebrated "sinusoid in → sinusoid out" property. This is why you hear about Fourier transforms more often than, for instance, Hadamard transforms.

- Provocatively speaking, linearity is the more "important" property, relative to time-invariance. Although the frequency response notion requires time-invariance, the impulse response idea can still be used even if the system is not time-invariant as long as its linear, except you now need a big set of impulse responses for different times instead of just one global impulse response. The superposition trick still applies; people will sometimes use the term "time-varying convolution" (although that is a rather awkward and likely misleading name) for something like this:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau,$$

where $h(t,\tau)$ is represents the response to an impulse $\delta(t-\tau)$.

Chapter 4

More on Continuous-Time Convolution

We begin by recalling the definition of an *impulse response*. If the input to a system is an impulse; i.e., $x(t) = \delta(t)$, then we call the corresponding output the *impulse response* and denote it as $y(t) = h(t)$. We can define an impulse response for systems that are neither linear nor time-invariant, but this concept is most useful for LTI systems. The remainder of this chapter assumes that all systems are LTI.

4.1 The convolution integral

We first considered continuous time convolution in the previous chapter and will revisit it here. Remember, *by definition* if $x(t) = \delta(t)$ (i.e., an impulse), then the output is $h(t)$, the impulse response. Now suppose the input is a scaled and delayed impulse,

$$x(t) = A\delta(t - t_0), \tag{4.1}$$

where the scaling factor is A and the time delay is t_0. We know from linearity and time-invariance that the output is a scaled and delayed version of the impulse response,

$$y(t) = Ah(t - t_0). \tag{4.2}$$

But what we really want to investigate is the output for a completely arbitrary input, a completely general function $x(t) = f(t)$. Let's start to build up this function from impulses by first considering multiplying $f(t)$ by a time-shifted impulse; that is, let $x(t) = f(t)\delta(t - \tau)$. By the sampling property we have $x(t) = f(\tau)\delta(t - \tau)$. By linearity and time invariance, the output of the system is $y(t) = f(\tau)h(t - \tau)$.

We can now construct our input signal from an integral over τ of these weighted impulses by noticing that the sifting property allows us to exactly construct $f(t)$ in this manner:

$$x(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t - \tau)d\tau = f(t). \tag{4.3}$$

Since integration is a linear operation, the output can be determined by applying exactly the same integral in τ:

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t - \tau)d\tau. \tag{4.4}$$

This is the *convolution integral* and is possibly the most important equation of this course.

Here we have expressed it in terms of an arbitrary input $x(t) = f(t)$, but it is usually expressed in terms of simply $x(t)$,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \tag{4.5}$$

The convolution operator is usually taken to be the asterisk and the above equation is thus written,

$$y(t) = x(t) * h(t). \tag{4.6}$$

Looking more closely inside the convolution integral, we see that the second function is $h(t-\tau)$, but it is a function of τ, not t, since τ is the variable of integration; t is whatever time we want to consider. By focusing on τ, we can write $h(t-\tau) = h(-(\tau - t))$. By writing it this way, it is now clear that with respect to τ, $h(t-\tau)$ is *flipped* (as a result of the initial negative sign) and then *shifted* by t, either to the left (for negative t) or the right (for positive t).

4.2 Properties of convolution

The following properties of convolution are very helpful for solving problems.

1. **Commutativity**

   Commutativity states that

   $$x(t) * h(t) = h(t) * x(t). \tag{4.7}$$

   It can readily be proved by making the change of variable $\lambda = t - \tau$ in the convolution integral:

   $$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{+\infty}^{-\infty} x(t-\lambda)h(\lambda)(-d\lambda) \\ &= \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda \\ &= h(t) * x(t). \end{aligned} \tag{4.8}$$

   Commutativity is a very useful property because it lets us choose which function to flip and shift.

2. **Associativity**

   Convolution is associative in that

   $$[x(t) * y(t)] * z(t) = x(t) * [y(t) * z(t)]. \tag{4.9}$$

   It can be proved by writing out both of the double integrals (one on each side of the equation) and making appropriate changes in variables until they are of the same form.

3. **Distributivity**

   Convolution is distributive with respect to addition:

   $$x(t) * [y(t) + z(t)] = x(t) * y(t) + x(t) * z(t). \tag{4.10}$$

   It is easily proved by distributing the sum of functions inside the convolution integral.

4. **Time shift**

   The time shift property of convolution states that if the inputs are shifted, the output is shifted by the sum of the input shifts. If $y(t) = x(t) * h(t)$, then

$$x(t - t_1) * h(t - t_2) = y(t - t_1 - t_2). \tag{4.11}$$

5. **Differentiation**

   The differentiation property states that if either of the inputs are differentiated, the output is also differentiated. If $y(t) = x(t) * h(t)$, then

$$\frac{dx(t)}{dt} * h(t) = x(t) * \frac{dh(t)}{dt} = \frac{dy(t)}{dt}. \tag{4.12}$$

6. **Convolution with an Impulse**

   As a consequence of the sifting property of the delta function, any function convolved with an impulse is itself:

$$x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau = x(t). \tag{4.13}$$

   When combined with the time shift property, it is clear that a shifted impulse is a delay operator:

$$x(t) * \delta(t - t_0) = x(t - t_0). \tag{4.14}$$

4.3 Convolution examples

The following examples range from very simple ones that can be done by inspection to a rather complicated convolution that is facilitated by a series of graphs (a.k.a. *graphical convolution*).

For each example, we are computing $y(t) = h(t) * x(t)$.

1. $x(t) = u(t)$ and $h(t) = \delta(t - 5)$

   By inspection, note that the delayed impulse simply shifts the step function, resulting in

$$y(t) = u(t) * \delta(t - 5) = u(t - 5). \tag{4.15}$$

2. $x(t) = u(t)$ and $h(t) = [\delta(t + 1) - \delta(t - 1)]$

   Convolve with each delta function separately and then combine:

$$y(t) = u(t + 1) - u(t - 1). \tag{4.16}$$

3. $x(t) = u(t)$ and $h(t) = u(t)$

   For this problem it doesn't matter which function we flip and shift since $x(t) = h(t)$. The convolution integral becomes:

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)u(t - \tau)d\tau. \tag{4.17}$$

The first step function can be handled by noting that all it does is change the lower limit of integration to zero:

$$y(t) = \int_{0}^{+\infty} u(t-\tau)d\tau. \tag{4.18}$$

The remaining step function is one for $\tau < t$ and is zero for $\tau > t$. Thus the integral is zero for $t < 0$ and is the following for $t > 0$:

$$y(t) = \int_{0}^{t} 1d\tau = \tau\Big|_{\tau=0}^{\tau=t} = t. \tag{4.19}$$

Rather than stating the result in terms of two separate ranges of t, it can be conveniently written as

$$y(t) = u(t) * u(t) = tu(t). \tag{4.20}$$

So the convolution of two steps is a ramp.

4. $x(t) = u(t-3)$ and $h(t) = \exp(-2t)u(t)$

This problem can also be handled by writing out the convolution integral, but now there is the choice of which function to flip and shift. Either choice will yield the correct answer; here we choose to flip and shift $h(t)$:

$$y(t) = \int_{-\infty}^{+\infty} u(\tau-3)\exp(-2(t-\tau))u(t-\tau)d\tau. \tag{4.21}$$

Both step functions can be handled similarly to the previous example by noting that the first one changes the lower limit of integration to three and the second one changes the upper limit of integration to t.

$$y(t) = \int_{3}^{t} \exp(-2(t-\tau))d\tau. \tag{4.22}$$

Now the term $\exp(-2t)$ can be brought out of the integral since it is not a function of τ, and the integral can be readily completed:

$$\begin{aligned} y(t) &= \exp(-2t)\int_{3}^{t}\exp(2\tau)d\tau \\ &= \exp(-2t)\frac{1}{2}\exp(2\tau)\Big|_{\tau=3}^{\tau=t} \\ &= \frac{1}{2}\exp(-2t)[\exp(2t)-\exp(6)] \\ &= \frac{1}{2}[1-\exp(-2(t-3))]. \end{aligned} \tag{4.23}$$

Recalling that $y(t) = 0$ for $t < 3$, we can express $y(t)$ as,

$$y(t) = \frac{1}{2}[1-\exp(-2(t-3))]u(t-3). \tag{4.24}$$

See Figure 4.1 for a plot of $y(t)$.

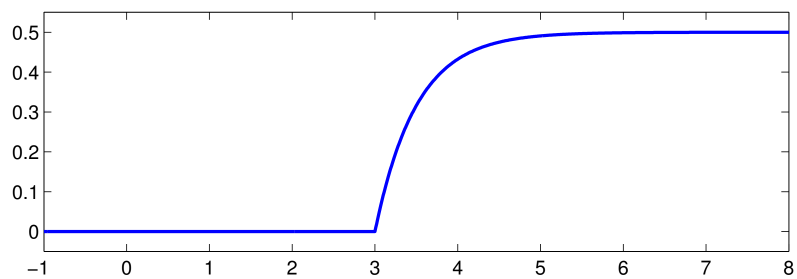


Figure 4.1: Convolution result $y(t)$ for Example 4.

5. $x(t) = 2(t-1)[u(t-1) - u(t-2)]$ and $h(t) = [u(t) - u(t-2)]$

   This problem can best be handled by what we refer to as **graphical convolution**. We first plot $x(t)$ and $h(t)$ vs. t as shown in the top plot of Figure 4.2. Here we choose to flip and shift $x(t)$ in the convolution integral. Recalling that we can write $x(t-\tau)$ as $x(-(\tau - t))$, it is clear that $x(t-\tau)$ is *flipped* with respect to τ and then *shifted* by t. The second plot in the figure shows $x(t-\tau)$ and $h(\tau)$ for a value of $t < 0$. We can see that $x(t-\tau)$ is non-zero from $\tau = t-2$ to $\tau = t-1$, and that in this range we have $x(t-\tau) = t - \tau - 1$ (we just substituted $t-\tau$ for t in the expression for $x(t)$).

   Now we must perform the convolution integral for various values of t. It is convenient to think of the point t as a "handle" that is being used to slide $x(t-\tau)$ to various places on the τ axis. The limits of the convolution integral are the range of overlap for the two functions; it is usually much easier to visualize this range graphically than to figure out where shifted and flipped step functions are non-zero. For example, if $t < 1$, there is no overlap at all and the convolution integral is zero; we have called that "Region 1" in the figure and $y_1(t) = 0$.

   If we continue to slide t to the right along the τ axis, the two functions begin to overlap when $t-1 = 0$, or $t = 1$. This region of partial overlap is Region 2 in the figure, and you can see that the functions overlap only between $\tau = 0$ and $\tau = t-1$; these values of τ become the range of integration. We have,

$$\begin{aligned} y_2(t) &= \int_0^{t-1} 2(t-\tau-1)d\tau \\ &= 2(t\tau - 0.5\tau^2 - \tau)\Big|_{\tau=0}^{\tau=t-1} \\ &= 2t(t-1) - (t-1)^2 - 2(t-1) \\ &= 2t^2 - 2t - t^2 + 2t - 1 - 2t + 2 \\ &= t^2 - 2t + 1 \\ &= (t-1)^2. \end{aligned} \quad (4.25)$$

   This result is valid for $1 \le t < 2$, which is the extent of Region 2.

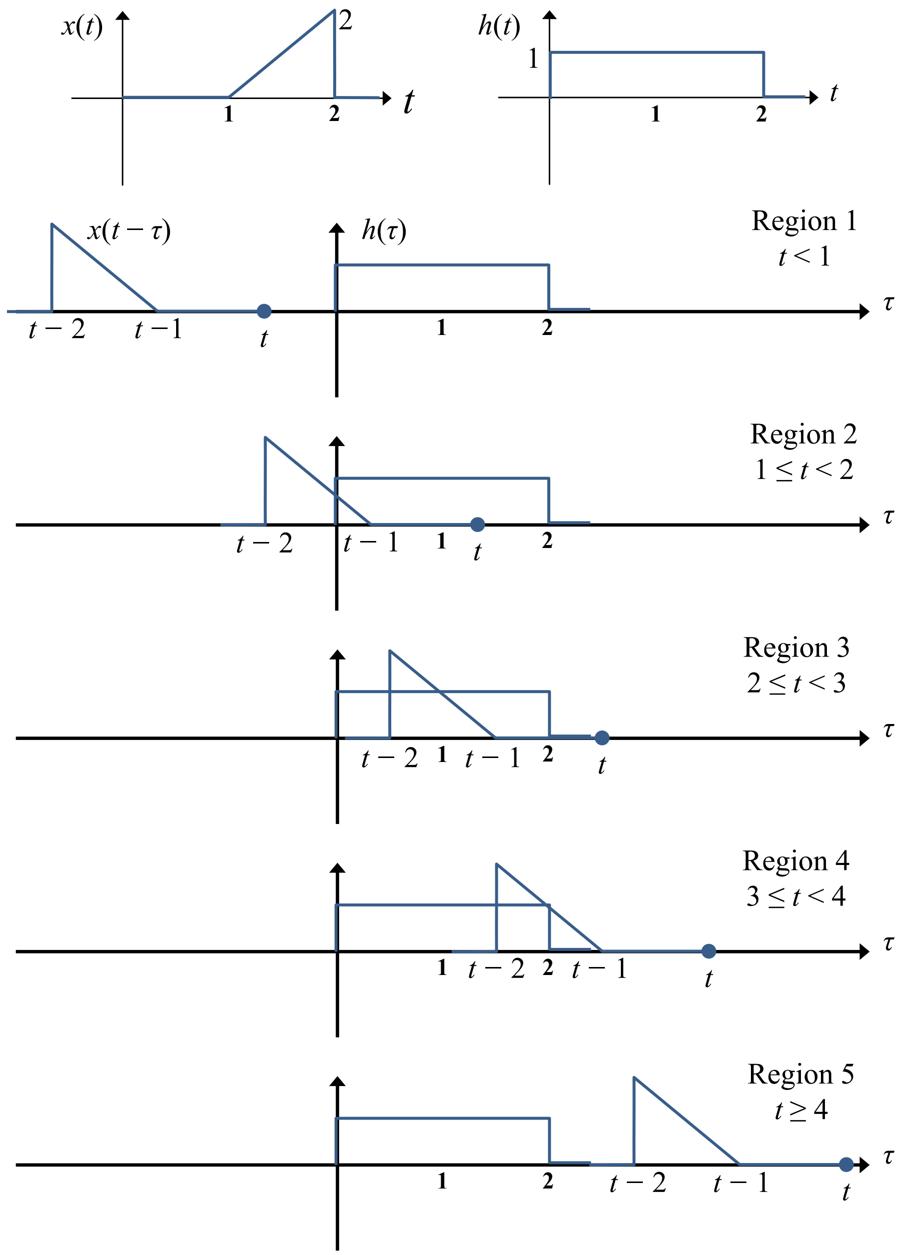


Figure 4.2: Graphical convolution steps for Example 5.

For $2 \le t < 3$ (Region 3), $x(t-\tau)$ completely overlaps with $h(\tau)$, and the limits of integration are from $t-2$ to $t-1$:

$$\begin{aligned}
y_3(t) &= \int_{t-2}^{t-1} 2(t-\tau-1)d\tau \\
&= 2(t\tau - 0.5\tau^2 - \tau)\Big|_{\tau=t-2}^{\tau=t-1} \\
&= 2t(t-1) - (t-1)^2 - 2(t-1) - 2t(t-2) + (t-2)^2 + 2(t-2) \\
&= 2t^2 - 2t - t^2 + 2t - 1 - 2t + 2 - 2t^2 + 4t + t^2 - 4t + 4 + 2t - 4 \\
&= 1.
\end{aligned} \tag{4.26}$$

Despite the somewhat messy integral, the answer is very simple. It can actually be seen by inspection that the integral is simply the area of the triangle, which is just 1. So we really didn't even have to integrate.

For $3 \le t < 4$ (Region 4), $x(t-\tau)$ only partially overlaps with $h(\tau)$ as it continues to slide to the right, and the limits of integration are from $t-2$ to 2:

$$\begin{aligned}
y_4(t) &= \int_{t-2}^{2} 2(t-\tau-1)d\tau \\
&= 2(t\tau - 0.5\tau^2 - \tau)\Big|_{\tau=t-2}^{\tau=2} \\
&= 4t - 4 - 4 - 2t(t-2) + (t-2)^2 + 2(t-2) \\
&= 4t - 8 - 2t^2 + 4t + t^2 - 4t + 4 + 2t - 4 \\
&= -t^2 + 6t - 8 \\
&= -(t-3)^2 + 1.
\end{aligned} \tag{4.27}$$

The last region is for $t > 4$, where $x(t-\tau)$ no longer has any overlap with $h(\tau)$. Thus, $y_5(t) = 0$.

To summarize, we have:

$$y(t) = \begin{cases} 0 & t < 1 & \text{Region 1} \\ (t-1)^2 & 1 \le t < 2 & \text{Region 2} \\ 1 & 2 \le t < 3 & \text{Region 3} \\ -(t-3)^2 + 1 & 3 \le t < 4 & \text{Region 4} \\ 0 & 4 \le t & \text{Region 5} \end{cases} \tag{4.28}$$

We can perform a reasonableness check by testing for continuity between each region. In Region 2, $y_2(1) = 0$ and $y_2(2) = 1$, so it is clearly continuous with Regions 1 and 3. Similarly, for Region 4 we can see that $y_4(3) = 1$ and $y_4(4) = 0$, so it is continuous with Regions 3 and 5. The following figure is a plot of $y(t)$ for all regions, again showing continuity. It is possible to have discontinuous convolution results, but only when one or both of the input signals contains delta functions.

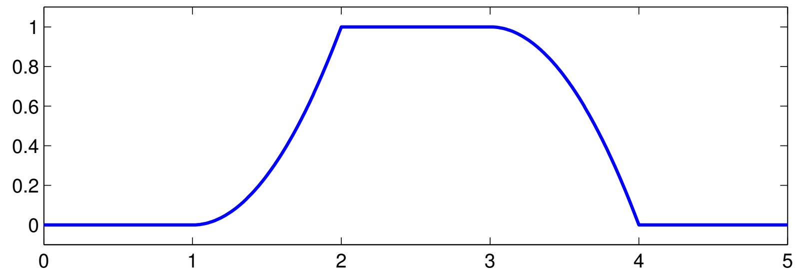


Figure 4.3: Convolution result $y(t)$ for Example 5.

4.4 Some final comments

Although it may not be not self-evident, developing convolution intuition is useful. Since almost everything of engineering interest can be approximated in some way as an LTI system, having a sense of what types of inputs result in what types of outputs is an important part of overall engineering intuition and experience. This intuition is best developed by graphical convolution, either performed manually (as in Example 5 above), or by using appropriate software (e.g., the continuous convolution GUI). Here are a few items that you should observe as you work through specific problems:

- When convolving simple shapes, convolution tends to make signals smoother (recall Example 3 where convolving two steps produced a ramp). That isn't always true when convolving really complicated signals, but we won't be doing those manually.

- When convolving two signals of finite length, the length of the output is the sum of the lengths of the inputs.

- When convolving two signals that don't start at $t = 0$, the output starts at the sum of the start times of the two input signals (actually this is true whether or not they start at $t = 0$).

- Choose carefully which signal to flip and shift. It is usually easier to flip and shift the "simpler" signal (perhaps the one that isn't shifted to start with, or the one with the simpler mathematical representation). Unfortunately it isn't always obvious which one is "simpler". Note that in Example 5, we picked the "wrong" signal to flip and shift; the integrals would have been slightly easier had we flipped and shifted the rectangular pulse instead of the triangular one, but the final answer would be the same.

- Convolving two rectangular pulses will yield either a triangular pulse if they are of the same length or a trapezoidal pulse if they are of different lengths. This type of convolution is best done graphically without doing any integration (i.e, find the region boundaries, calculate the output at the transition points from the overlapped area, and connect the dots).

Chapter 5

Cross-Correlation and Matched Filtering

[THIS CHAPTER NEEDS MANY MANY MANY FIGURES]

As we saw in the last chapter, when two signals are convolved, one of them is flipped and then slides across the other signal as per the convolution integral:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{5.1}$$

A very similar and closely related operation, called "cross-correlation," consists of simply sliding one signal by the other *without* flipping it first. Unfortunately, there is no commonly accepted notation for cross-correlation like there is with the centered asterisks for convolution.[1] We will notate correlation using the capital letter R with subscripts:

$$R_{sf}(t) = \int_{-\infty}^{\infty} s(\tau - t)f(\tau)d\tau. \tag{5.2}$$

As an aid to keep track of which signal is which, the "slider" comes first and is named $s(t)$, and the "fixed" signal comes second and is named $f(t)$.

Autocorrelation is the cross-correlation of a signal with itself, e.g., $R_{ss}(t)$. Autocorrelation can be visualized as sliding a signal across itself.

[1] Another common notation is to use a five-sided star, $R_{sf}(t) = s(t) \star f(t)$, or less commonly, a circle with a cross, $R_{sf}(t) = s(t) \otimes f(t)$. We find that students often confuse the six-sided star of convolution with the five-sided star, and hence stay away from the $\star$ notation. We also shy away from the $\otimes$ notation, since that symbol is more often used to denote a Kronecker matrix product.

5.1 Cross-correlation properties

Cross-correlation is related to convolution as follows:

$$\begin{aligned} R_{sf}(t) &= \int_{-\infty}^{\infty} f(\tau)s(\tau - t)d\tau && (5.3) \\ &= \int_{-\infty}^{\infty} f(\tau)s(-(t-\tau))d\tau && (5.4) \\ &= f(t) * s(-t) && (5.5) \\ &= s(-t) * f(t). && (5.6) \end{aligned}$$

In words, cross-correlation is the convolution of the two signals after time-reversing the one that slides, $s(t)$.

Cross-correlation is *not* commutative, which is clear from its relation to convolution.

$$R_{sf}(t) = s(-t) * f(t) \neq s(t) * f(-t). \quad (5.7)$$

Since $s(t) * f(-t) = R_{fs}(t)$, cross-correlation is not commutative.

However, if $s(t)$ is even (that is, $s(t) = s(-t)$), then cross-correlation is commutative, which follows directly from its relation to convolution.

The cross-correlation integral is often seen written in a different form, which we derive here by a change of variable from τ to $\lambda = \tau - t$:

$$\begin{aligned} R_{sf}(t) &= \int_{-\infty}^{\infty} s(\tau - t)f(\tau)d\tau && (5.8) \\ &= \int_{-\infty}^{\infty} s(\lambda)f(\lambda + t)d\lambda && (5.9) \\ &= \int_{-\infty}^{\infty} s(\tau)f(\tau + t)d\tau. && (5.10) \end{aligned}$$

This form, which is commonly found in textbooks, suggests another way to think about cross-correlation. In the original way, we visualized it as fixing $f(t)$ and sliding $s(t)$ from left to right. In this second representation, we are fixing $s(t)$ and sliding $f(t)$ from right to left. They are equivalent, but the original way is more similar to graphical convolution.

The autocorrelation is always even,

$$\begin{aligned} R_{ss}(t) &= s(t) \star s(t) && (5.11) \\ &= s(-t) * s(t) && (5.12) \\ &= s(t) * s(-t) && (5.13) \\ &= R_{ss}(-t). && (5.14) \end{aligned}$$

The autocorrelation always achieves its maximum value at $t = 0$,

$$R_{ss}(0) = \int_{-\infty}^{\infty} [s(t)]^2 dt. \quad (5.15)$$

Figure 5.1: Correlation classifier.

Although cross-correlation is not commutative, $R_{sf}(t)$ is related to $R_{fs}(t)$:

$$\begin{aligned} R_{fs}(t) &= \int_{-\infty}^{\infty} f(\tau - t)s(\tau)d\tau && (5.16) \\ &= \int_{-\infty}^{\infty} f(\lambda)s(\lambda + t)d\lambda && (5.17) \\ &= \int_{-\infty}^{\infty} f(\tau)s(\tau + t)d\tau && (5.18) \\ &= R_{sf}(-t). && (5.19) \end{aligned}$$

Reversing the order of the functions when cross-correlating time-reverses the result.

5.2 Cross-correlation examples

The following examples range from quite easy to rather complicated.

For each example, we are computing $R_{sf}(t)$.

1. $s(t) = \delta(t) - \delta(t-1)$ and $f(t) = u(t)$

   Work this one as $s(-t) * f(t)$.

$$s(-t) = \delta(t) - \delta(t+1) \quad (5.20)$$

$$\begin{aligned} s(-t) * f(t) &= [\delta(t) - \delta(t+1)] * u(t) && (5.21) \\ &= u(t) - u(t+1) && (5.22) \\ &= -[u(t+1) - u(t)]. && (5.23) \end{aligned}$$

   This problem could also be worked graphically by sliding $s(t) = \delta(t) - \delta(t-1)$ by $u(t)$.

5.3 Matched filter implementation

One of the primary applications of cross-correlation is *matched filtering*. A matched filter is an LTI system that implements cross-correlation where the signal $s(t)$ is a known template signal and $f(t)$ is the input to the filter. Since an LTI system is completely described by its impulse response $h(t)$, an LTI system that implements cross-correlation must have $h(t) = s(-t)$. The output is:

$$y(t) = h(t) * f(t) = s(-t) * f(t) = R_{sf}(t). \quad (5.24)$$

The correlator classifier computation described in (5.41) has the form of a multiplier, followed by an integrator (Figure 5.1). There is an alternative way of implementing this operation that consists of running the data through an LTI filter, and then sampling the output of the filter at a particular time. Let us reconsider that inner product operation:

$$\int_{-\infty}^{\infty} x(t)s_k^*(t)dt. \quad (5.25)$$

Define an LTI filter with the impulse response $h_k(t) = s_k^*(-t)$. Notice that $h_k(t)$ is a conjugated and time-reversed version of the template; this is the impulse response of the "matched filter." If we feed this filter the data, the output is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_k(t-\tau)d\tau. \tag{5.26}$$

Notice that the output of the filter at time 0 is equivalent to the inner product of (5.25):

$$y(0) = \int_{-\infty}^{\infty} x(\tau)h_k(0-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h_k(-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)s_k^*(\tau)d\tau. \tag{5.27}$$

5.4 Delay estimation

This matched filter viewpoint of Section 5.3 is particularly handy if we are looking for a time-shifted version of the template and we don't know the amount of the shift. Quite often, we may have a single "template," and the amount of the time shift (usually a delay) is what we are interested in learning. In many radar and sonar applications, we transmit a pulse and wait for it to bounce off of an object and return. The time delay is proportional to the range to the object. The radar and sonar signals are often "bandpass" signals with a high-frequency carrier, and hence well represented by complex baseband representations. This is the main reason we made sure the exposition in Section 5.7 worked for complex signals, and not just real signals.

Instead of considering a set of templates $s_1(t)$, $s_2(t)$, etc., suppose the data is described by $x(t) = s(t-\Delta) + n(t)$, where Δ is some unknown delay. We can imagine running an infinite set of correlation detectors, with each detector corresponding to a particular Δ. Fortunately, we do not actually need to build an infinite number of multipliers and an infinite number of integrators. If we neglect noise, the output of the matched filter in this scenario is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} s(\tau-\Delta)h(t-\tau)d\tau & (5.28) \\ &= \int_{-\infty}^{\infty} s(\tau-\Delta)s^*(-(t-\tau))d\tau = \int_{-\infty}^{\infty} s(\tau-\Delta)s^*(\tau-t)d\tau. & (5.29) \end{aligned}$$

This is a good time to introduce the Schwarz inequality, which, in this context, says

$$\left| \int_{-\infty}^{\infty} f(t)g^*(t)dt \right|^2 \leq \left[\int_{-\infty}^{\infty} |f(t)|^2 dt \right] \left[\int_{-\infty}^{\infty} |g(t)|^2 dt \right] \tag{5.30}$$

with equality if and only if $f(t) = \alpha g(t)$, for some α. If we correspond $f(t)$ with $s(\tau-\Delta)$ and $g(t)$ with $s(\tau-t)$, we see that

$$\left| \int_{-\infty}^{\infty} s(\tau-\Delta)s^*(\tau-t)d\tau \right|^2 \tag{5.31}$$

is going to be largest when $s(\tau-\Delta) = s^*(\tau-t)$, i.e. when $t = \Delta$. This gives us a procedure for finding the time-shift: filter the data with a conjugated, reverse copy of transmitted waveform, and find the t for which the energy at the output power of the filter is the largest.

The output of the matched filter for $\Delta = 0$ has a special name; it is called the *autocorrelation* of the signal:

$$\int_{-\infty}^{\infty} s(\tau)s^*(\tau-t)d\tau. \tag{5.32}$$

Figure 5.2: Example of a linear FM sweep.

Figure 5.3: Autocorrelation function of the linear FM sweep in Figure 5.2.

If we substitute $\tilde{\tau} = \tau - t$, giving $\tau = \tilde{\tau} + t$, we can rewrite the autocorrelation of $s(t)$ as

$$\int_{-\infty}^{\infty} s(\tilde{\tau} + t)s^*(\tilde{\tau})d\tilde{\tau}, \tag{5.33}$$

which is how you will see it defined in most textbooks. The autocorrelation of a waveform is basically that waveform convolved with a time-reversed version of itself. We can also define a more general cross-correlation between two functions:

$$\int_{-\infty}^{\infty} x(\tilde{\tau} + t)s^*(\tilde{\tau})d\tilde{\tau}, \tag{5.34}$$

In a simple radar ranging system, if the data contains a single "target," the output of the matched filter (not including noise) is the autocorrelation function of the transmitted waveform, time-shifted to where the target is located in time.

The autocorrelation is an important tool in waveform design for range estimation, since it characterizes many aspects of system performance. The curvature of the peak of the autocorrelation is related to how accurately we can estimate the time shift, and hence range, under noisy conditions. If multiple targets are present, targets that are close in range may blur together and look like a single target. The broadness of the peak of the autocorrelation is indicative of the *resolution* of the system, which addresses how well we can discriminate between close targets. The sidelobes of the autocorrelation function provide a sense of how a target with weak reflectivity may "hide" in the sidelobe of a target with stronger reflectivity. This brief paragraph is intended to only provide a small taste of such issues; thorough definitions of these properties and related explorations are best found in dedicated texts on remote sensing.

Matched filtering also allows radar designers to pull off a slick trick called *pulse compression*. To obtain good range resolution, we might intuitively want our waveforms to be short in duration. But transmitters are usually limited in terms of output power, which leads us to want to use long waveforms to put lots of energy on the target to be able to combat noise. If we can find a waveform with a lengthy time extent, but whose autocorrelation has a narrow mainlobe, match filtering essentially compresses the energy over that long time extent into a narrow pulse in the matched filter output, so we can enjoy both good resolution and good signal-to-noise ratio. For instance, a linear FM waveform (Figure 5.2) has an autocorrelation function that looks somewhat sinc-like (Figure 5.3), although it is by no means a "pure" sinc.

5.5 Causal concerns

In most applications, $s(t)$ will be non-zero for some time range $0 \leq t \leq L$, for some length L, and be zero outside of that range. This means that the matched filter $h(t) = s(-t)$, as defined in the sections above, will be non-causal and hence be unable to be implemented in real-time systems. In practice, this is not cause for much concern. We can implement a *causal* matched filter $h_c(t) = s^*(-(t - L)) = s^*(L - t)$, in which we simply shift our original matched filter $h(t)$ to the right by enough time that the resulting $h_c(t)$ is causal. All of our previous results apply, except that the outputs are now delayed by L.

5.6 A caveat

The "derivations" of correlation classifiers and matched filter structures given above were somewhat heuristic, since they were intended to be digestible by readers lacking experience with probability theory and random processes. These structures are usually derived via a more detailed analysis that includes rigorously modeling the noise, in which one can make precise statements about the signal-to-noise ratio.

5.7 Under the hood: squared-error metrics and correlation processing

Suppose we measured a waveform that had the form of a signal corrupted by additive noise

$$x(t) = s_k(t) + n(t), \tag{5.35}$$

where s_1, s_2, etc. represent different kinds of signals, which we call *templates*, that we are trying to discriminate between. We are not going to say much about the noise; properly treating it requires a thorough discussion of probability and random processes, which is beyond the scope of this text.

One reasonable approach might be to measure the "error" between each template and the actual measured data, and pick the template that yields the lowest error. The *squared error* is commonly employed:

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t) - s_k(t)|^2 dt &= \int_{-\infty}^{\infty} [x(t) - s_k(t)][x(t) - s_k(t)]^* dt & (5.36)\\
&= \int_{-\infty}^{\infty} [x(t) - s_k(t)][x^*(t) - s_k^*(t)] dt & (5.37)\\
&= \int_{-\infty}^{\infty} x(t)x^*(t) - s_k(t)x^*(t) - s_k^*(t)x(t) - s_k(t)s_k^*(t) dt & (5.38)\\
&= \int_{-\infty}^{\infty} |x(t)|^2 - s_k(t)x^*(t) - s_k^*(t)x(t) - |s_k(t)|^2 dt. & (5.39)
\end{aligned}$$

The squared error is mathematically convenient. It is particularly appropriate if the noise $n(t)$ is Gaussian, but you do not need to know anything about Gaussian probability distributions – or probability in general – to understand what follows.

For convenience, suppose the signal templates are all normalized to have the same energy, i.e. $\int_{-\infty}^{\infty} |s_k(t)|^2 dt$ is the same for every k. Also, note that the $\int_{-\infty}^{\infty} |x(t)|^2$ term does not depend upon k. So in trying to minimize the error with respect to k, we can drop the first and last terms in (5.39). The middle two terms consist of something added to its complex conjugate, so our minimization problem reduces to finding the k that minimizes

$$2\Re e \left\{ - \int_{-\infty}^{\infty} x(t) s_k^*(t) \right\}, \tag{5.40}$$

or equivalently, finding the k that maximizes

$$\Re e \left\{ \int_{-\infty}^{\infty} x(t) s_k^*(t) \right\} dt. \tag{5.41}$$

We dropped the 2 in front since it does not change the result of the maximization. This procedure is referred to as a *correlation classification*; it may be the most common form of basic "template matching" used in pattern recognition. Essentially, we want to take the inner product[2] of the data with each template, and find the template that produces the "best" match.[3]

[2]A more mathematically thorough treatment would demonstrate that this inner product is an aspect of the "projection" of the data onto the template.

[3]We put "best" in quotes since other kinds of error functions could be used that might have a different idea of what "best" means.

Chapter 6

Review of Fourier Series

Before we present Fourier transforms in general, we will review Fourier series, since they provide an intuitive springboard from which we can later study Fourier transform, which extend the idea of Fourier series to signals that are not periodic.

6.1 Fourier synthesis sum and analysis integral

Recall that a continuous-time signal $x(t)$ with fundamental period T_0 can be written (assuming some technical conditions that hold for most signals of interest in engineering) as a sum of weighted complex sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(j\frac{2\pi}{T_0}kt\right) = \sum_{k=-\infty}^{\infty} a_k \exp\left(j\omega_0 kt\right). \tag{6.1}$$

where the second form arises from writing the fundamental frequency in radians as $\omega_0 = 2\pi/T_0$. The ω_0 is slightly compact, but the form with $2\pi/T_0$ often makes potential cancelations in various computations more obvious. In either form, this summation is referred to as the **Fourier synthesis sum** since we are synthesizing a periodic signal from a sum of sinusoids.

If $x(t)$ consisted of sums or products of a few sinusoids, one can often find the a_k directly by expanding such terms using Euler's formulas. In more difficult cases, we are forced to invoke the **Fourier analysis integral**:

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\exp\left(-j\frac{2\pi}{T_0}kt\right)dt = \frac{1}{T_0}\int_{T_0} x(t)\exp\left(-j\omega_0 kt\right)dt. \tag{6.2}$$

The subscript T_0 indicates that the integral can be taken over any interval of length T_0. Common choices include $-T_0/2$ to $T_0/2$ and 0 to T_0, but something like $-T_0/4$ to $3T_0/4$ is also feasible. Depending on how $x(t)$ is defined, some choices of integration interval may be more or less convenient than others. One must be careful to not forget to include the $1/T_0$ constant in front. If you work a problem and find a T_0 in your answer for a_k, it sometimes means that you forgot the $1/T_0$.

Recall that if you only need the "D.C." or "average" value a_0, it is often convenient to plug 0 in for k right at the beginning, since that greatly simplifies the integration:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\exp\left(-j\frac{2\pi}{T_0}0t\right)dt = \frac{1}{T_0}\int_{T_0} x(t)dt.$$

Also, recall that if $x(t)$ is real, the Fourier series coefficients satisfy the conjugate symmetry $a_{-k} = a_k^\star$. Using that fact, along with the inverse Euler's formula for a cosine, we can write the Fourier series representation of a real $x(t)$ in terms of cosines:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos\left(\frac{2\pi}{T_0}kt + \angle\{a_k\}\right). \tag{6.3}$$

6.2 System response to a periodic signal

In a previous chapter, after viewing the idea of frequency response for discrete-time systems, we introduced the idea of frequency response for continuous-time systems. If the input to an LTI system with frequency response $H(j\omega)$ is $\exp(j\omega_0 kt)$, then the output is $H(j\omega_0 k)\exp(j\omega_0 kt)$. If we can write an input as a Fourier series with Fourier coefficients a_k, we can just multiply each coefficient by the frequency response of the system evaluated at the appropriate frequency:

$$y(t) = \sum_{k=-\infty}^{\infty} H(j\omega_0 k)a_k \exp(j\omega_0 kt) = \sum_{k=-\infty}^{\infty} b_k \exp(j\omega_0 kt), \tag{6.4}$$

where $b_k = H(j\omega_0 k)a_k$, and one could replace ω_0 with $2\pi/T_0$ if desired.

If the input $x(t)$ is real, we can write the output as a sum of cosines:

$$\begin{aligned} y(t) &= H(j0)a_0 + \sum_{k=1}^{\infty} |H(j\omega_0 k)| 2a_k \cos(\omega_0 kt + \angle\{a_k\} + \angle\{H(j\omega_0 k)\}) & (6.5) \\ &= b_0 + \sum_{k=1}^{\infty} 2|b_k| \cos(\omega_0 kt + \angle\{b_k\}). & (6.6) \end{aligned}$$

6.3 Properties of Fourier series

There are a number of properties of Fourier series that are useful for obtaining Fourier coefficients without integration. Start by assuming that the Fourier series coefficients for a signal $x(t)$ are a_k and those of a second signal $w(t)$ are b_k, and that both $x(t)$ and $w(t)$ have the same fundamental period T_0.

1. **Linearity**

   If $y(t) = Ax(t) + Bw(t)$, then it also has a period of T_0 and its Fourier coefficients are $c_k = Aa_k + Bb_k$. Linearity is readily proven by simple substitution and the proof is not shown here.

2. **Scaling and Offset**

   Let $y(t) = Ax(t) + C$. This is a special case of linearity for which the second signal is a simple D.C. offset; its Fourier coefficients are zero for $k \neq 0$, and $b_0 = C$. The scale factor affects all of the Fourier coefficients whereas the offset affects only the $k = 0$ (D.C.) term. Thus,

$$\begin{aligned} c_k &= Aa_k \quad \text{for } k \neq 0 \\ c_0 &= Aa_0 + C. & (6.7) \end{aligned}$$

3. **Time Shift**

   If $y(t) = x(t - t_d)$, then it also has a period of T_0 and its Fourier coefficients are $c_k = a_k \exp(-jk\omega_0 t_d)$. Note that $b_0 = a_0$ since $k = 0$.

   The proof is as follows:

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_0^{T_0} x(t - t_d)\exp(-jk\omega_0 t)dt \\
&= \frac{1}{T_0}\int_{t_d}^{t_d+T_0} x(\tau)\exp(-jk\omega_0(\tau + t_d))d\tau \quad \text{[substitute } \tau = t - t_d\text{]} \\
&= \frac{1}{T_0}\int_0^{T_0} x(\tau)\exp(-jk\omega_0(\tau + t_d))d\tau \quad \text{[can integrate over any period]} \\
&= \exp(-jk\omega_0 t_d)\frac{1}{T_0}\int_0^{T_0} x(\tau)\exp(-jk\omega_0\tau)d\tau \\
&= a_k \exp(-jk\omega_0 t_d).
\end{aligned} \tag{6.8}$$

4. **Time Reversal (Flip)**

   If $x(t)$ is real and $y(t) = x(-t)$ (i.e., it is flipped in time), then its Fourier coefficients are $c_k = a_k^\star$. The proof is as follows:

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_0^{T_0} x(-t)\exp(-jk\omega_0 t)dt \\
&= \frac{1}{T_0}\int_0^{-T_0} x(\tau)\exp(jk\omega_0\tau)(-d\tau) \quad \text{[substitute } \tau = -t\text{]} \\
&= \frac{1}{T_0}\int_0^{T_0} x(\tau)\exp(jk\omega_0\tau)d\tau \\
&= a_k^\star.
\end{aligned} \tag{6.9}$$

6.4 Fourier series of a symmetric "square wave"

Consider a periodic function with fundamental period T_0 that is defined over one period as

$$x(t) = \begin{cases} 1 & \text{for } T_0/4 \le t < T_0/4 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } -T_0/2 \le t < T_0/2. \tag{6.10}$$

We can find the Fourier series coefficients via:

$$a_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} \exp\left(-j\frac{2\pi}{T_0}kt\right) dt \tag{6.11}$$

$$= \frac{1}{T_0}\frac{1}{-j\frac{2\pi}{T_0}k} \exp\left(-j\frac{2\pi}{T_0}kt\right)\Bigg|_{t=-T_0/4}^{t=T_0/4} \tag{6.12}$$

$$= \frac{1}{j2\pi k}\left[\exp\left(j\frac{\pi}{2}k\right) - \exp\left(-j\frac{\pi}{2}k\right)\right] \tag{6.13}$$

$$= \frac{1}{\pi k}\sin\left(\frac{\pi}{2}k\right). \tag{6.14}$$

Since k is an integer, $\sin(\pi k/2)$ can only take on three values:

$$\sin\left(\frac{\pi}{2}k\right) = \begin{cases} 0 & \text{for even } k \\ 1 & \text{for } k = \ldots, -7, -3, 1, 5, 9, \ldots \\ -1 & \text{for } k = \ldots, -9, -5, -1, 3, 7, \ldots \end{cases} \tag{6.15}$$

This implies that the even harmonics are missing, which is what gives the square wave its hollow, clarinet-like tone.

The $k = 0$ case of (6.14) is problematic since both the numerator and denominator are zero. Applying L'Hopitals rule[1] by taking the derivative of the numerator and the derivative of the denominator yields

$$a_0 = \lim_{k\to 0}\frac{1}{\pi k}\sin\left(\frac{\pi}{2}k\right) = \frac{\lim_{k\to 0}\frac{\pi}{2}\cos\left(\frac{\pi}{2}k\right)}{\lim_{k\to 0}\pi} = \frac{1}{2}. \tag{6.16}$$

Taking a limit as $k \to 0$ is a bit unsettling since k is "supposed" to be an integer. We take solace in noticing that we get the same answer in we use the D.C. trick:

$$a_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} 1 dt = \frac{T_0/2}{T_0} = \frac{1}{2},$$

which makes sense when you look at the waveform.

6.4.1 Lowpass filtering the square wave

Suppose we ran the symmetric square wave of Sec. 6.4 through a "brickwall" lowpass filter with a cutoff frequency ω_c:

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases} \tag{6.17}$$

One can quibble over whether the $\le$ should be a strict inequality; we will avoid coming up with situations where it would matter. In any case, Section 7.7.2 will illustrate that it is impossible to construct such a "brickwall" filter in real life anyway; it is just a convenient approximation.

[1]Calculus professors are fond of coming up with examples that look safe but violate various conditions of L'Hopitals rule, which leads to incorrect results when carelessly applied. When reading this text, you can take it on faith that such conditions are satisfied and L'Hopitals rule is perfectly safe.

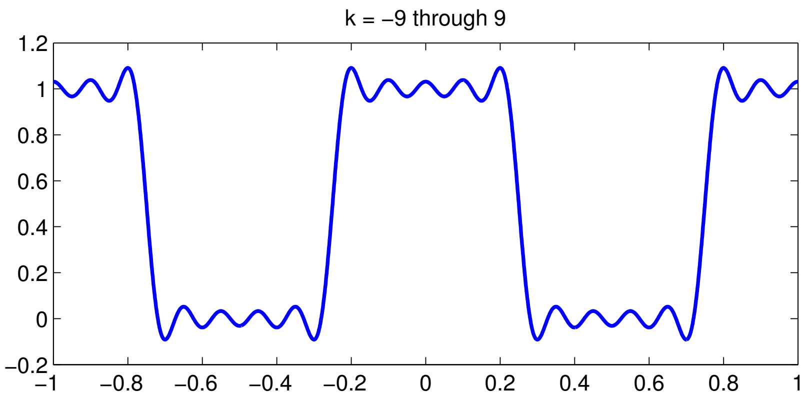


Figure 6.1: A low-pass filtered square wave with terms up through $k = \pm 9$.

We found that the Fourier coefficients of this square wave were

$$a_k = \frac{1}{\pi k} \sin\left(\frac{\pi}{2} k\right). \tag{6.18}$$

Suppose the filter has a cutoff of $\omega_c = 4\omega_0$. Since the square wave has no even harmonics, we don't have to worry about the cutoff falling right on a harmonic. (Again, this is rarely an issue in practice).

Only the D.C. term and the first and third harmonics survive going through the filter. Note that $a_1 = 1/\pi$ and $a_3 = -1/(3\pi)$. Since $x(t)$ is real, the Fourier series coefficients have conjugate symmetry, which is particularly simple in this case since a_1 and a_3 are real. Hence, the output can be written as

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 t) - \frac{2}{3\pi} \cos(3\omega_0 t).$$

Note you could rewrite the last term as $+\frac{2}{3\pi} \cos(3\omega_0 t + \pi)$.

You will see things like this in lab when you look at square waves on the oscilloscope; besides limitations of the bandwidth of your signal generator and the bandwidth of your scope, various bits of stray capacitance in your cable connections act like a lowpass filter.

Figure 5.1 illustrates a square wave with $T_0 = 1$ ($\omega_0 = 2\pi$) that has been filtered with a cutoff frequency of $\omega_c = 10\omega_0$. This filter retains six frequency components, corresponding to D.C. and the odd harmonics ($k = 0$, $k = \pm 1$, $k = \pm 3$, $k = \pm 5$, $k = \pm 7$, and $k = \pm 9$). Although this signal is recognizable as a square wave, you can see obvious overshoot and undershoot at each edge; this is referred to as the "Gibbs phenomenon" and is always evident whenever any discontinuous signal is lowpass filtered.

6.5 What makes Fourier series tick?

Section (3.4) noted that complex sinusoids of the form $\exp(j\omega t)$ (from which we can easily build real sines and cosines) are special because they are "eigenfunctions" of LTI systems; if you put one into a system, the magnitude and phase of the sinusoid may change, but the frequency stays the same. The frequency response of the system plays the role of an "eigenvalue."

Sinusoids whose frequencies are all multiples of some fundamental frequency have another interesting property called *orthogonality.* This extends the notion of vectors in physics being at a "90 degree angle" to functions defined over some period. Two different functions are orthogonal if they produce zero when integrated "against" each other; roughly speaking, this can be thought of as projecting one vector onto another. One quirk of dealing with complex numbers is that one of the functions, when put into the integral, needs to be complex conjugated.

To explore orthogonality, let us first compute this integral:

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \exp\left(j\frac{2\pi}{T_0}kt\right) dt = \frac{1}{T_0}\,\frac{1}{j\frac{2\pi}{T_0}k}\exp\left(j\frac{2\pi}{T_0}kt\right)\Bigg|_{t=-T_0/2}^{t=T_0/2} \tag{6.19}$$

$$= \frac{1}{j2\pi k}[\exp(j\pi k) - \exp(-j\pi k)] \tag{6.20}$$

$$= \frac{1}{\pi k}\sin(\pi k). \tag{6.21}$$

This equals 0 for $k \neq 0$. The $k = 0$ case is a bit trickier; taking the derivative of the denominator and numerator of (6.21) to apply L'Hopitals rule yields:

$$\lim_{k\to 0}\frac{1}{\pi k}\sin(\pi k) = \frac{\lim_{k\to 0}\pi\cos(\pi k)}{\lim_{k\to 0}\pi} = 1. \tag{6.22}$$

If we plug $k = 0$ into (6.19) right at the beginning, we get the same answer, since we wind up integrating the constant 1 over a length of T_0. Our results shouldn't seem too surprising; remember that complex exponentials are really sinusoids, and if you integrate any sinusoid over a period, the upper hump cancels the lower hump. Another important observation is that we could have chosen any interval of length T_0 to integrate over in (6.21).

That integral in (6.21) comes in handy when checking for orthogonality:

$$\frac{1}{T_0}\int_{T_0}\exp\left(j\frac{2\pi}{T_0}kt\right)\exp\left(-j\frac{2\pi}{T_0}\ell t\right) dt = \frac{1}{T_0}\int_{T_0}\exp\left(j\frac{2\pi}{T_0}[k-\ell]t\right) dt \tag{6.23}$$

$$= \delta[k-\ell]. \tag{6.24}$$

Now that we have developed this notion of orthogonality, we can try plugging a Fourier series into a Fourier analysis integral to see if everything checks out:

$$\frac{1}{T_0}\int_{T_0}\left[\sum_{k=-\infty}^{\infty} a_k \exp\left(j\frac{2\pi}{T_0}kt\right)\right]\exp\left(-j\frac{2\pi}{T_0}\ell t\right) dt \tag{6.25}$$

$$= \sum_{k=-\infty}^{\infty} a_k\left[\frac{1}{T_0}\int_{T_0}\exp\left(j\frac{2\pi}{T_0}\ell t\right)\exp\left(-j\frac{2\pi}{T_0}kt\right) dt\right] \tag{6.26}$$

$$= \sum_{k=-\infty}^{\infty} a_k\delta[\ell-k] = a_\ell. \tag{6.27}$$

This shows that the Fourier analysis integral does what it is supposed to—it knows how to extract Fourier series coefficients.

6.6 Under the hood

The Gibbs phenomenon which arises in the case of discontinuous functions should make us look upon the equals sign in (6.1) with skepticism.

When we write equations like

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(j\omega_0 k t\right),$$

with infinite summations, we really mean something like

$$x(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} a_k \exp\left(j\omega_0 k t\right).$$

But what does *that* really mean? For continuous $x(t)$, it is not too mysterious. But for the general case of Fourier series representations of discontinuous $x(t)$, it means something like

$$\lim_{N\to\infty} \int_T \left| x(t) - \sum_{k=-N}^{N} a_k \exp\left(j\omega_0 k t\right) \right|^2 dt = 0.$$

For a particular value of t, you can think of that absolute square as measuring the error at that point t. As you increase the number of the terms in the Fourier series summation, the summation gets "closer" to $x(t)$, but it does so in a particular way that involves the *average error* over a period going to zero. The Gibbs peaks do not mess that up since although they never go away, they get narrower and narrower.

The upshot of all this is that sometimes there is some fine print hiding under our equal signs. That is about all we will have to say about that level of gorey mathematical detail; digging into such details is typically at the heart of graduate classes with names like "measure theory" and "functional analysis."

Chapter 7

Fourier Transforms

7.1 Motivation

Take another look at the Fourier analysis integral and Fourier series summation:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \exp\left(-j\frac{2\pi}{T_0}kt\right) dt, \tag{7.1}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(j\frac{2\pi}{T_0}kt\right). \tag{7.2}$$

We have always thought of $x(t)$ as being a periodic signal with periodic T_0. But Fourier series are also useful for describing finite-length signals; for instance, in communication theory, signals in the class of functions $x : [0, T_0] \to \mathbb{C}$ are often of interest. From this viewpoint, we are not really worried what the Fourier series for $x(t)$ is outside of that interval of interest. Interestingly, $[0, T_0]$ is an uncountably infinite set, but we claim that $x(t)$ can be described by a countably infinite set of coefficients. We get away with this since practical functions of interest in engineering satisfy various arcane rules that $x(t)$ must obey for the Fourier series representation to work. (Mathematicians can cook up perverse functions that do not obey these rules; we will leave that to them.)

Consider functions in an equivalent class $x : [-T_0/2, T_0/2] \to \mathbb{C}$. We will explore what happens as $T_0 \to \infty$, i.e, the period grows to infinity, and make some appropriate modifications to the above formulas. Some of these modifications seem will seem unmotivated; they are unlikely to be things you might think of until you try to make this heuristic derivation work, get close to the end, and then realize that you need some some tweaks.

Notice that as the period becomes longer, the spacing between adjacent frequencies in the complex exponential (e.g. between k and $k + 1$) become smaller. This leads us to suspect that we could treat the summation as a Riemann sum and ultimately replace it with an integral. This intuitive approach might provide insight into how someone might come up with the concept of Fourier transforms. A more rigorous approach would involve a more modern approach to calculus called Lebesgue integration, but you will not need to worry about that unless you dig more into the theory in graduate school.

To prepare for this journey, multiply the Fourier analysis integral by T_0:

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x(t) \exp\left(-j\frac{2\pi}{T_0}kt\right) dt. \tag{7.3}$$

Also, dress up the the Fourier series by putting $2\pi/T_0$ at the end of the equation and $T_0/(2\pi)$ split up near the beginning (so that we are not actually changing the equation):

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} T_0 a_k \exp\left(j\frac{2\pi}{T_0}kt\right) \frac{2\pi}{T_0}. \tag{7.4}$$

Now wave your hands vigorously while letting $\omega = \lim_{T_0\to\infty} 2\pi k/T_0$, $d\omega = \lim_{T_0\to\infty} 2\pi/T_0$, and $X(j\omega) = \lim_{T_0\to\infty} T_0 a_k$.

After the dust settles, we are left with the classic Fourier transform pair:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt, \tag{7.5}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \exp(j\omega t)\, d\omega. \tag{7.6}$$

What you just experienced was not anything like a rigorous proof; we hope it was fun anyway.

7.2 A key observation

Notice that the inverse Fourier transform basically rewrites a function as a sum of complex exponentials; the ideas underlying our previous discussion about inputting a Fourier series into an LTI system apply here as well. If a signal with Fourier transform $X(j\omega)$ is input to a system with a frequency response of $H(j\omega)$, the output of the system is

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)X(j\omega) \exp(j\omega t)\, d\omega. \tag{7.7}$$

Noticing that (7.7) looks like an inverse Fourier transform, we see one of the most celebrated properties of Fourier transforms: convolution in the time domain, e.g. $y(t) = h(t) * x(t)$, corresponds to multiplication in the frequency domain, e.g. $Y(j\omega) = H(\omega)X(\omega)$. Note that if you had not already heard about convolution being commutative, you would immediately realize that it has to be since multiplication is commutative. Also, notice that this property holds for functions in general; $H(\omega)$ does not really need to be a frequency response of some system and $X(\omega)$ does not really need to be the Fourier transform of an input. We will squeeze gallons of juice out of this property. In particular, for LTI systems in a cascade configuration, we can swap the order of the systems, and we can also theoretically replace a cascade of two systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$ with a single system with frequency response $H(j\omega) = H_1(j\omega)H_2(j\omega)$.

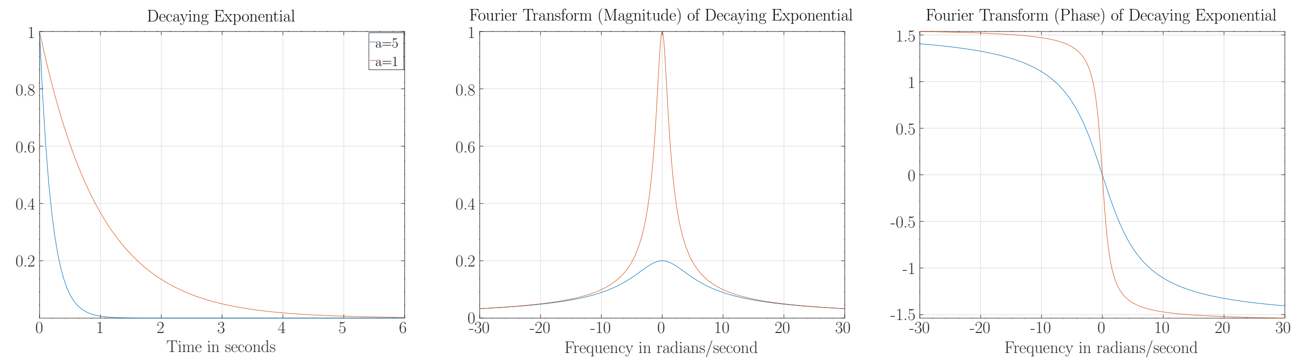


Figure 7.1: Fourier transforms of decaying exponentials. Red and blue lines are for $a = 1$ and $a = 5$, respectively.

7.3 Your first Fourier transform: decaying exponential

Let us find the Fourier transform of $x(t) = \exp(-at)u(t)$ for $a > 0$.

$$X(j\omega) = \int_{-\infty}^{\infty} \exp(-at)u(t)\exp(-j\omega t)dt = \int_{0}^{\infty} \exp(-at)\exp(-j\omega t)dt \tag{7.8}$$

$$= \int_{0}^{\infty} \exp(-[a+j\omega]t)dt = \frac{1}{-(a+j\omega)}\exp(-[a+j\omega]t)\Big|_{0}^{\infty} \tag{7.9}$$

$$= \frac{1}{a+j\omega}. \tag{7.10}$$

Note that $a > 0$ ensures that $\exp[(-a + j\omega)t]$ is a decaying exponential; we need that so that the upper limit goes to zero.[1] Figure 7.1 shows some examples. There are many examples of physical systems that have the impulse response $h(t) = C\exp(-at)u(t)$; for instance, it characterizes response of an RC circuit with the capacitor going to ground, the resistor going to an input voltage, and the output voltage measured between the capacitor and the resistor. The frequency response of this system is the Fourier transform we just computed, times a constant.

Sometimes we can use the following trick to easily compute the magnitude of a frequency response:

$$|H(j\omega)|^2 = H(j\omega)H^{\star}(j\omega) = \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{a-j\omega}\right) = \frac{1}{a^2+\omega^2}, \tag{7.11}$$

$$|H(j\omega)| = \sqrt{\frac{1}{a^2+\omega^2}}. \tag{7.12}$$

This is a "single-pole" lowpass filter; we will talk about what we mean by "pole" in later chapters. The "DC gain" is $H(j0) = 1/a$. One particularly useful description of such a system is the frequency at which

[1] In later chapters, we will look at a relative of the Fourier transform called the Laplace transforms, which can be used to analyze the $a \geq 0$ case.

the power of the input signal is cut by half relative to the power at DC; for the moment, let us call it ω_c:

$$\frac{1}{2}|H(j0)|^2 = |H(\omega_c)|^2, \tag{7.13}$$

$$\frac{1}{2} \cdot \frac{1}{a^2} = \frac{1}{a^2 + \omega_c^2}. \tag{7.14}$$

By inspection, the half-power cutoff is $\omega_c = a$.

The phase of the frequency response is

$$H(j\omega) = -\arctan(\omega_c/a). \tag{7.15}$$

7.3.1 Frequency response example

Suppose a continuous-time LTI system has the impulse response $h(t) = \exp(-0.1t)u(t)$, and it is fed an input signal $x(t) = 7\cos(\sqrt{3}(0.1)t + \pi/5)$.

The magnitude of the frequency response of the system, evaluated at the input frequency, is

$$|H(j\sqrt{3}(0.1))| = \sqrt{\frac{1}{(0.1)^2 + [\sqrt{3}(0.1)]^2}} = \sqrt{\frac{1}{(0.1)^2[1+3]}} = 5. \tag{7.16}$$

And the corresponding phase is

$$H(j\sqrt{3}(0.1)) = -\arctan[\sqrt{3}(0.1)/0.1] = -\arctan[\sqrt{3}] = -\pi/3. \tag{7.17}$$

Notice we set up the numbers so that they invoke a 30-60-90 triangle; real life is rarely so convenient.

To find the output, the amplitude of the input wave is multiplied by the magnitude of the frequency response, and the phase of the input wave is added to the phase of the frequency response:

$$y(t) = 5 \times 7\cos(\sqrt{3}(0.1)t + \pi/5 - \pi/3) = 35\cos(\sqrt{3}(0.1)t - 2\pi/15). \tag{7.18}$$

7.4 Your first Fourier transform property: time shift

In practice, computing a Fourier transform by explicitly evaluating the Fourier transform integral is a last resort. For most functions of interest in typical engineering applications, someone will have already worked out a basic Fourier transform pair that is pretty close to what you need. If you can find a transform that close to what you need in a table somewhere, you can often apply a some Fourier transform properties to massage the Fourier transform pair you found into the Fourier transform pair you need. These properties are generally stated along the lines of "if $X(j\omega)$ is the Fourier transform of $x(t)$, then..."

We will only bother to prove a few of them here; we will state many without proof. Once you see a few proofs, you will be able to do the others, since they all follow a similar patterns. Suppose $X(j\omega)$ is the Fourier transform of $x(t)$. Then the Fourier transform of the shifted function $y(t) = x(t - t_0)$ is

$$Y(j\omega) = \int_{-\infty}^{\infty} x(t - t_0)\exp(-j\omega t)\, dt \tag{7.19}$$

$$= \int_{-\infty}^{\infty} x(\tau)\exp(-j\omega[\tau + t_0])\, d\tau = \int_{-\infty}^{\infty} x(\tau)\exp(-j\omega\tau)\exp(-j\omega t_0)\, d\tau \tag{7.20}$$

$$= \exp(-j\omega t_0)\int_{-\infty}^{\infty} x(\tau)\exp(-j\omega\tau)\, d\tau = \exp(-j\omega t_0)\, X(j\omega). \tag{7.21}$$

In the first step above, we made the substitution $\tau = t - t_0$; we have $t = \tau + t_0$ and $dt = d\tau$. We see that a shift in the time domain corresponds to multiplication by a complex sinusoid in the frequency domain.

Many proof involving transforms follow this approach: you plug something into one of the transform integrals (forward or inverse), you might or might not do a change of variables, and then you rearrange the resulting expression so that an integral recognizable as a transform jumps out at you.

7.5 Your second Fourier transform: delta functions

The Fourier transform of $x(t) = \delta(t - t_0)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) \exp(-j\omega t) dt = \exp(-j\omega t_0). \tag{7.22}$$

The special case of $x(t) = \delta(t)$ yields $X(j\omega) = 1$. This is amazing—somehow an uncountably infinite sum of a uncountably infinite number of sinusoids, at every possible frequency, each with equal amplitude, adds up to give you a delta function! Most textbooks just state this fact and move on. Meditate upon this for a second. First of all, notice that $X(j\omega) = X(-j\omega)$ for this $x(t) = \delta(t)$ case; looking at the inverse Fourier transform, this tells you that $\delta(t)$ can be written as a sum of cosines. Each one of these cosines is 1 at $t = 0$, but then they all "wave" differently outside of $t = 0$. Somehow all of the cosines cancel each other out for $t \neq 0$, but they all add up at $t = 0$ to create something infinite at that point.

7.5.1 Sanity check

Let us pause to make sure that (7.22) is consistent with the shifting property we derived in Sec. 7.4.

Recall that $x(t) * \delta(t - t_0) = x(t - t_0)$; convolving a function with a Dirac delta function just shifts that function.

The convolution property of Fourier transforms tells us that the Fourier transform of $x(t)*\delta(t-t_0) = x(t-t_0)$ will be the Fourier transform of $x(t)$ times the Fourier transform of $\delta(t - t_0)$, namely $X(j\omega) \exp(-j\omega t_0)$. This is consistent with the shifting property we derived earlier.

7.6 Your second Fourier transform property: derivatives in time

If we take the derivative of both sides of the definition of the inverse Fourier transform (7.6) with respect to t, we find

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\exp(j\omega t)\, d\omega\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}\{\exp(j\omega t)\}\, d\omega \tag{7.23}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)[j\omega\exp(j\omega t)]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(j\omega)]\exp(j\omega t)\, d\omega. \tag{7.24}$$

The last expression looks like the Fourier transform of $j\omega X(j\omega)$. Hence, taking a derivative with respect to time in the time domain corresponds to multiplication by $j\omega$ in the frequency domain.

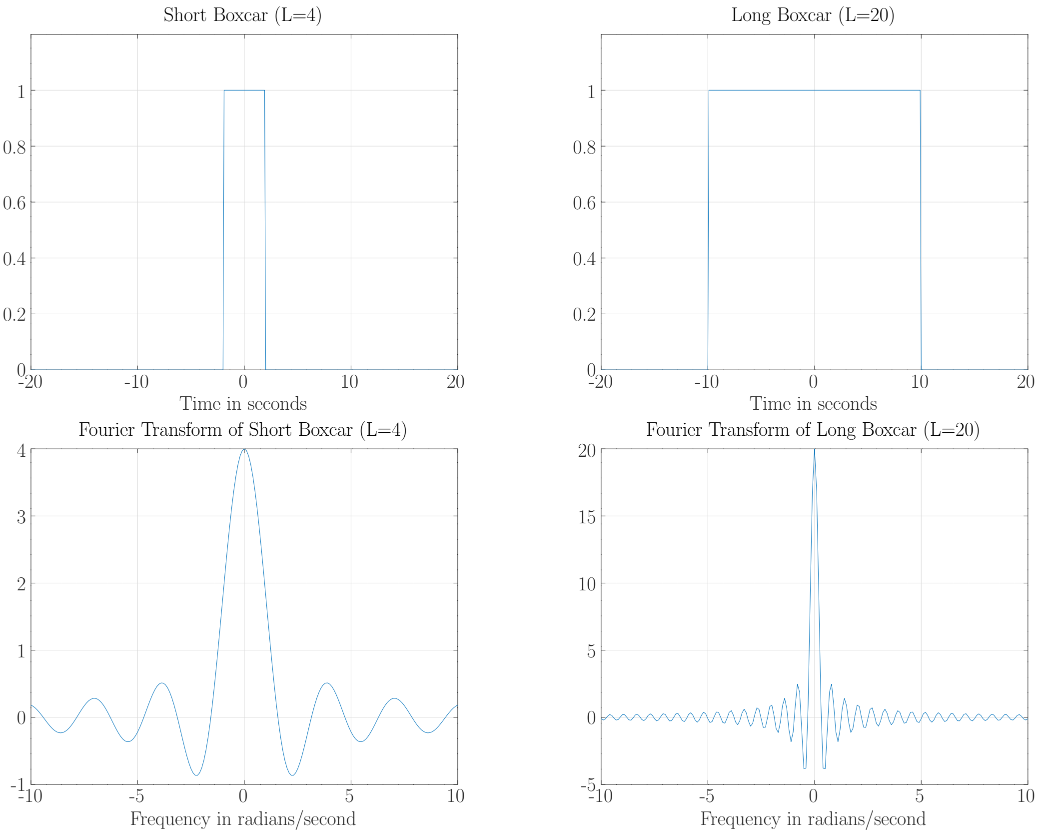


Figure 7.2: The bottom row shows the Fourier transforms of the time-domain functions on the top row. Since the sinc functions on the right are real-valued, its reasonable to simply plot the function, with the graphs running both positive and negative along the vertical axis. With complex-valued functions, it is usually clearer to plot the magnitude and phase separately.

7.7 Rectangular boxcar functions

7.7.1 Fourier transform of a symmetric rectangular boxcar

Consider a pulse of unit height of length L, centered at the origin, $x(t) = u(t+L/2) - u(t-L/2)$. Its Fourier transform is

$$\begin{aligned} X(j\omega) &= \int_{-L/2}^{L/2} \exp(-j\omega t)\, dt = \frac{1}{-j\omega} \exp(-j\omega t)\Big|_{t=-L/2}^{t=L/2} && (7.25) \\ &= \frac{1}{j\omega}\left[\exp\left(j\frac{L}{2}\omega\right) - \exp\left(-j\frac{L}{2}\omega\right)\right] = \frac{2}{2j\omega}\left[\exp\left(j\frac{L}{2}\omega\right) - \exp\left(-j\frac{L}{2}\omega\right)\right] && (7.26) \\ &= \frac{2}{\omega}\sin\left(\frac{L}{2}\omega\right) = \frac{\sin(L\omega/2)}{\omega/2}. && (7.27) \end{aligned}$$

Concerning the indeterminate $\omega = 0$ case, L'Hopital's rule says

$$X(j0) = \lim_{\omega \to 0} \frac{2}{\omega} \sin\left(\frac{L}{2}\omega\right) = 2\frac{\lim_{\omega \to 0} \frac{L}{2} \cos\left(\frac{L}{2}\omega\right)}{\lim_{\omega \to 0} 1} = L. \tag{7.28}$$

Remember the trick for finding $X(j0)$ without much work:

$$X(j0) = \int_{-\infty}^{\infty} x(t) \exp(-j0t)\, dt = \int_{-\infty}^{\infty} x(t) dt. \tag{7.29}$$

Applying this to the boxcar results in something consistent with what we found from L'Hopital rule:

$$x(t) = \int_{-L/2}^{L/2} 1 dt = L. \tag{7.30}$$

The zero crossings of $X(j\omega)$ occur when $L\omega/2 = k\pi$, i.e. $\omega = 2\pi k/L$, where k is a nonzero integer.

This kind of sin(something)/something expression is called a "sinc" function. Many textbooks and computer programs (such as MATLAB) use the notation sinc($\cdot$); unfortunately, different books and programs use different definitions of sinc. To avoid confusion, we will explicitly "spell out" what we mean when writing equations. Figure 7.2 shows a short boxcar transforming into a wide sinc function, and a longer boxcar transforming into a narrower sinc function.

In your previous studies, you may have looked at the frequency response of a discrete-time running sum; it looks similar to our sinc function here, but instead of having the form sin(something)/something, it had the form sin(something)/sin(something); this gives the discrete-time version a repeating structure. Sometimes this discrete-time response is called a "sind" function – you will also hear it referred to as an "aliased sinc" or "digital sinc" (although both of those terms are perhaps misleading).

The boxcar is an easily created signal – for instance, in an electrical circuit context, one might just flip a voltage or current on and then back off using a mechanical or electronic switch of some kind. Later, we will see how boxcars also provide a convenient "window" for defining limited-time versions of more complicated functions.

Recall that the first Fourier transform we looked at was the transform of a decaying exponential (Sec. 7.3). We used it as an example of a frequency response of a system, since continuous-time systems with decaying exponential impulse responses are ubiquitous. This "boxcar" function is a different story; you would be hard pressed to find a real-world system with a boxcar impulse response "in the wild," although professors like to put them on homework problems regardless of how unrealistic they might be. (This contrasts with discrete-time systems, in which the equivalent boxcar is easy to construct with a tiny microprocessor with sufficient memory.)

7.7.2 Inverse Fourier transform of single symmetric boxcar

We now explore the other direction. Consider an $X(j\omega)$ which is 1 between $-\omega_c$ and ω_c and 0 elsewhere. (We will try to avoid situations where we might be worried about what $X(j\omega_c)$ at *exactly* $\omega = \pm\omega_c$). Do not forget to divide by 2π; that is a quite common error! Here we go:

$$x(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \exp(j\omega t)\, d\omega = \frac{1}{j2\pi t} \exp(j\omega t)\Big|_{\omega=-\omega_c}^{\omega=\omega_c} \tag{7.31}$$

$$= \frac{1}{2j\pi t} \left[\exp(j\omega_c t) - \exp(-j\omega_c t)\right] = \frac{1}{\pi t} \sin(\omega_c t). \tag{7.32}$$

We see that the inverse transform of a rectangular window in frequency is a sinc function in time. By now, you should be quite familiar with applying L'Hopital's rule to find that $x(0) = \omega_c/\pi$. We could also apply this general trick for easily finding $x(0)$ from a Fourier transform $X(j\omega)$:

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\exp(j0t)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega. \tag{7.33}$$

The zero crossings of $x(t)$ occur when $w_c t = k\pi$, i.e. $t = \pi k/\omega_c$, where k is a nonzero integer.

Although boxcars do not describe the impulse response of many realistic systems, they do describe the frequency response of many almost-realistic systems, or at least hoped-for systems. The "almost" and "hoped-for" hedges come from observing the corresponding sinc impulse response $h(t)$; it is not causal, and actually has infinite time extent in both directions! So, a perfect "brickwall" filter $H(j\omega)$ is not something you can actually build. But you can try to get close, and we can often get close enough that we will invoke the brickwall filter as a convenient approximation.

Example: Ramp Filters

The "ramp filter" is part of the "projection-slice theorem," which is often used in medical imaging applications such as X-ray computer-aided tomography. We mention this for context; a full discussion of the projection-slice theorem is usually part of a graduate course on image processing. In its application domain, the filter is applied to functions of *space*, not time, but to remain consistent with our usual notation, we will go ahead and stick with time and use the variable t.

The "ramp" part of the name refers to the shape of the frequency response, which is given by

$$H(j\omega) = \begin{cases} j\omega & \text{for } |\omega| < \omega_0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.34}$$

This is a boxcar in the frequency domain, namely $u(\omega + \omega_0) - u(\omega + \omega_0)$, multiplied by $j\omega$. Section 7.6 showed that multiplying by $j\omega$ in the frequency domain corresponds to taking a derivative in the time domain. The inverse Laplace transform of the boxcar is given by (7.32), so we can simply compute

$$h(t) = \frac{d}{dt}\left\{\frac{\sin(\omega_0 t)}{\pi t}\right\} = \frac{\pi t[\omega_0 cos(\omega_t)] - \pi\sin(\omega_0 t)}{(\pi t)^2} \tag{7.35}$$

$$= \frac{t\omega_0 cos(\omega_t) - \sin(\omega_0 t)}{\pi t^2} = \frac{\omega_0}{\pi t}\cos(\omega_0 t) - \frac{1}{\pi t^2}\sin(\omega_0 t). \tag{7.36}$$

Either formula on line (7.36) works; one is not necessarily clearer than the other.

7.7.3 Observations about our boxcar examples

Three observations before we move on:

1. We have seen that a boxcar in time transforms into a sinc function in frequency, and the sinc function in time transforms into a boxcar in frequency. That kind of near-symmetry shows up frequently, both in terms of transform pairs and in transform properties. This should not be surprising, since the forms of the forward and inverse Fourier transforms are so similar. We say "near-symmetry" since the dual forms usually have variations in signs and the presence (or not) of 2π somewhere.

2. We made a slight notational change in the way we defined our boxcars. Although the math of the transforms we took is basically the same, the time and frequency domain have different interpretations in practice. We think of pulses of having a length L and filters having a cutoff frequency ω_c. The difference in parameterization also made a difference in the look-and-feel of the transforms beyond sign flips and factors of 2π.

3. We took the transforms of pulses, which gave us the sinc functions. Suppose we had done it the other way around; imagine someone gave us a sinc function, but we did not know anything about its transform. If we plug a sinc function into the forward or inverse transform formulas, we get something horribly complicated that would be quite challenging to attack using the usual tricks of Freshman calculus. This is another common pattern—transforming function A into function B using the FT formulas may be easy, but brute-force transforming B into A using the FT formulas may be difficult or impossible, so having a table of such pairs – where you computed whatever the easiest one was, and then can just look in the table when you need to go the other direction – is quite helpful.

7.8 Fourier transforms of deltas and sinusoids

Remember that the Fourier transform of $x(t) = \delta(t - t_0)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) \exp(-j\omega t) dt = \exp(-j\omega t_0). \tag{7.37}$$

The inverse Fourier transform of $X(j\omega) = \delta(\omega - \omega_0)$ is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) \exp(j\omega t) d\omega = \frac{1}{2\pi} \exp(j\omega_0 t). \tag{7.38}$$

Because of the linearity of Fourier transforms, we can move the 2π and write this pair as

$$\exp(j\omega_0 t) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi\delta(\omega - \omega_0), \tag{7.39}$$

which has the intriguing special case

$$1 \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi\delta(\omega), \tag{7.40}$$

Using (7.39), the inverse Euler's formulas, and the linearity of Fourier transforms, we can readily find the Fourier transform of cosines and sines:

$$\cos(\omega_0 t) = \frac{1}{2}[\exp(j\omega_0 t) + \exp(-j\omega_0 t)], \tag{7.41}$$

$$\cos(\omega_0 t) \overset{\mathcal{F}}{\Longleftrightarrow} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0). \tag{7.42}$$

$$\sin(\omega_0 t) = \frac{1}{2j}[\exp(j\omega_0 t) - \exp(-j\omega_0 t)], \tag{7.43}$$

$$\sin(\omega_0 t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0) = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0). \tag{7.44}$$

7.9 Fourier transform of periodic signals

We know we can represent periodic signals as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(jk\omega_0 t\right). \tag{7.45}$$

We can take the Fourier transform of this Fourier series by applying the transform of a complex sinusoid derived earlier, as well as the linearity of Fourier transforms:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - k\omega_0\right). \tag{7.46}$$

Chapter 8

Modulation

This chapter presents a simplified story of how AM radio works. This is a practical application that will let us introduce a few new Fourier properties in a concrete context while also reviewing some of the Fourier theory we already covered.

8.1 Fourier view of filtering

As a warm-up, recall the property that convolution in the time domain corresponds to multiplication in the frequency domain:

$$x(t) * h(t) \stackrel{\mathcal{F}}{\Longleftrightarrow} X(j\omega)H(j\omega). \tag{8.1}$$

Recall that the cosine function $x(t) = \cos(\omega_0 t)$ has the Fourier transform $X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$. If we input this function into a system with the general impulse response $h(t)$, applying the convolution property easily yields the Fourier domain description of the output $y(t)$:

$$\begin{aligned} Y(j\omega) &= [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]H(j\omega) && (8.2) \\ &= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0). && (8.3) \end{aligned}$$

In a previous chapter, we found that the inverse Fourier transform of $\delta(\omega - \omega_0)$ is $\exp(j\omega_0 t)/(2\pi)$. Hence, the inverse transform of (8.3) is

$$y(t) = H(j\omega_0)\frac{\exp(j\omega_0 t)}{2} + H(-j\omega_0)\frac{\exp(-j\omega_0 t)}{2}. \tag{8.4}$$

If $h(t)$ is real-valued, it turns out that its Fourier transform is conjugate symmetric, i.e. $H(-j\omega) = H^*(j\omega)$ (this should not be surprising since we have seen that Fourier series coefficients have the same property). Hence, if $h(t)$ is real-valued, we can simplify (8.4) as

$$\begin{aligned} y(t) &= H(j\omega_0)\frac{\exp(j\omega_0 t)}{2} + H^*(j\omega_0)\frac{\exp(-j\omega_0 t)}{2} && (8.5) \\ &= H(j\omega_0)\frac{\exp(j\omega_0 t)}{2} + H^*(j\omega_0)\frac{\exp(-j\omega_0 t)}{2} && (8.6) \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle\{H(j\omega_0)\}). && (8.7) \end{aligned}$$

There is nothing new there; this is the classic "sinusoid in → sinusoid out" property of LTI systems from Section 3.4 that we have hammered on repeatedly (as in Section 7.3.1), just redone using our new shiny Fourier transform machinery. This section is only a consistency check.

8.1.1 Filtering by an ideal lowpass filter

Suppose we had an ideal "brickwall" lowpass filter with cutoff frequency ω_{co}, defined by

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega \leq \omega_{co} \\ 0 & \text{for } \omega > \omega_{co} \end{cases}. \tag{8.8}$$

In an earlier chapter, we saw that the corresponding impulse response $h(t)$ is a sinc function. (As usual, we will try to avoid worrying about what happens exactly at $\omega = \omega_{co}$; this is not a filter we can realistically build anyway, so it is not a question that will arise in practice). The response to a pure sinusoid $x(t)$ with frequency ω_o would simplify to

$$y(t) = \begin{cases} x(t) & \text{for } \omega_{co} \leq \omega_o \\ 0 & \text{for } \omega_{co} > \omega_o \end{cases}. \tag{8.9}$$

Again, this is not new; we are reviewing it because (a) it is always good to review and (b) we will use it in Section 8.3.

8.2 Modulation property of Fourier transforms

We have often employed the property that convolution in the time domain corresponds to multiplication in the frequency domain:

$$x(t) * h(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(j\omega)H(j\omega). \tag{8.10}$$

This property has a "dual" property, which says that multiplication in the time domain corresponds to convolution in the frequency domain, divided by 2π:

$$x(t)p(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi} X(j\omega) * P(j\omega). \tag{8.11}$$

The proof is like the other Fourier transform property proofs you have seen; we will not spell out the details here.

This is the first time we have presented convolution *in the frequency domain;* it is defined analogous to time-domain convolution:

$$X(j\omega) * P(j\omega) = \int_{-\infty}^{\infty} X(j\xi)P(j(\omega - \xi))d\xi. \tag{8.12}$$

We changed h to p since h usually corresponds to the impulse response of a filter, which is not usually how we want to interpret what we are now calling p. (Also, be careful – that 2π is easy to forget!)

We approach this as engineers, not mathematicians; the context of these properties is that our *real-world implementations* take place on the left hand sides of (8.10) and (8.11). We usually build filters that operate in the time domain, and we build multiplication circuits that multiply in the time domain; the *frequency-domain interpretations* of the systems we build are found on the right hand sides of (8.10) and (8.11). A design procedure might begin on the frequency-domain side, but the ultimate implementation usually happens on the time-domain side. (In systems using extensive digital signal processing, where algorithms such as the FFT can be exploited, there may be times where the "implementation" takes place on the right hand side, but this approach is fairly rare in purely analog computation systems.)

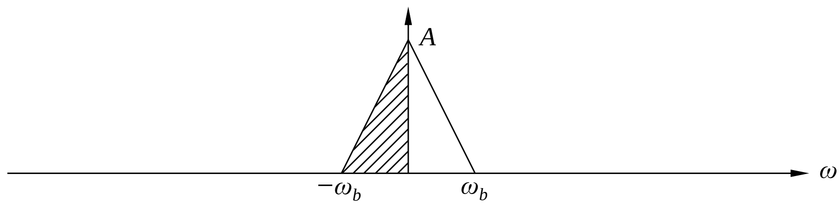


Figure 8.1: Conceptual Fourier transform of a "typical" real-valued bandlimited signal. The triangle is a "placeholder."

8.2.1 Modulation by a complex sinusoid

Suppose $p(t) = \exp(j\omega_0 t)$. In Section 7.8, we discovered that $P(j\omega) = 2\pi\delta(\omega - \omega_0)$. Inserting this into our modulation property tells us that

$$x(t)\exp(j\omega_0 t) \stackrel{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi} X(j\omega) * [2\pi\delta(\omega - \omega_0)] = X(j(\omega - \omega_0)).$$

You can also readily prove this by just plugging $X(j(\omega - \omega_0))$ into the inverse Fourier transform integral and doing a Freshman-calculus-style change of variable.

We see that multiplication by a complex sinusoid in the domain corresponds to a shift in the frequency domain. Communications engineers make tremendous use of this property to move signals around in the frequency domain.

Although it resembles the shift-in-time property we derived in Section 7.4,

$$x(t - t_0) \stackrel{\mathcal{F}}{\Longleftrightarrow} X(j\omega)\exp(-j\omega t_0),$$

you must be mindful of the differences (for instance, the difference of sign in the complex exponential).

8.3 Double Side Band Amplitude Modulation

Suppose we have a real-valued signal $x(t)$ that is bandlimited: $X(j\omega) = 0$ for $|\omega| \geq \omega_b$, that we want to transmit. We often draw such "generic" $X(j\omega)$ as triangles, with the left side shaded to keep track of conjugate frequency pairs, as shown in Figure 8.1. We do not expect the actual signal of interest to have a Fourier transform that is actually a triangle; we know that would be the square of a sinc function, which probably is never going to make the Billboard Top 100. It is just a shape traditionally used for bookkeeping purposes.

We are already fudging a little bit; no signal can be strictly time-limited and strictly band-limited. Because no one lives forever, the time constraint is always pressing. We will assume that our signals are "close enough" to being bandlimited that our analysis is a reasonable approximation.

8.3.1 DSBAM transmission

Earlier, we saw that multiplying a signal by a complex sinusoid shifts that signal in the frequency domain. Using this trick directly would require two channels of some kind, one to hold a "real" component and another to hold an "imaginary" component. We will revisit this in a later section, but for now, we would

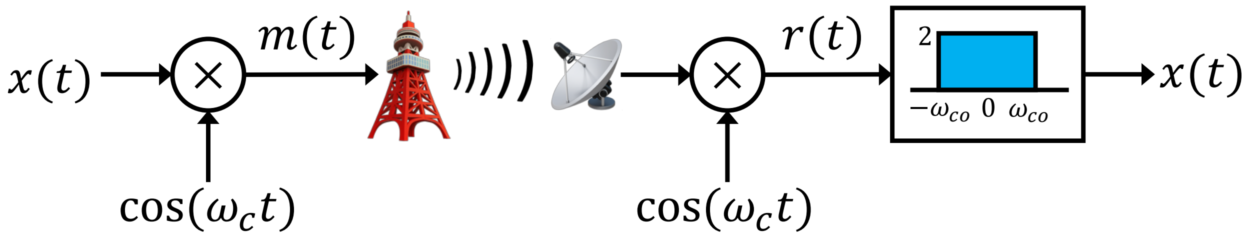


Figure 8.2: Block diagram of a DSBAM communication system.

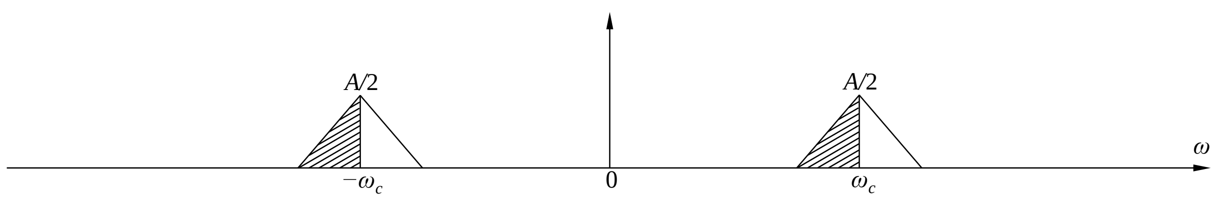


Figure 8.3: $M(j\omega)$, the Fourier transform of the modulated signal $m(t)$.

like to restrict ourselves to real-valued signals. Let us multiply $x(t)$ by by $\cos(\omega_c t)$, where ω_c is the called the carrier frequency, since we will think of the pure sinusoid $\cos(\omega_c t)$ as "carrying" the signal $x(t)$. Let us denote the *modulated signal* as $m(t) = x(t)\cos(\omega_c t)$, as shown in the left portion of Figure 8.2 The Fourier transform pairs and properties reviewed earlier give us:

$$M(j\omega) = \frac{1}{2\pi}X(j\omega) * [\pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)] \tag{8.13}$$

$$= \frac{1}{2}X(\omega - \omega_c) + \frac{1}{2}X(\omega + \omega_c), \tag{8.14}$$

as shown in Figure 8.3. The 1/2 might be vaguely interpreted as half of the energy in the spectrum of $x(t)$ landing on the "positive side" and the other half landing on the "negative side."

Communication engineers move signals in the frequency domain for two reasons: (1) so different radio stations, for instance, can broadcast over the same "airwaves" by choosing different carrier frequencies[1] and (2) we can encode our signal at a frequency for which radio waves have an easier time being generated and propagating in practice. If you listen to Clark Howard on WSB AM 750, that means Clark has been modulated to 750 kHz. His audio-frequency voice signal, if amplified and shoved out a realistically sized antenna, would not get very far. (The Georgia Tech radio station WREK is at 91.1 MHz, but it uses frequency modulation instead of AM. Information on FM radio is readily available from other sources.)

8.3.2 DSBAM reception

Upon receiving $m(t)$, how might we get $x(t)$ back? An initial thought might be to divide by $\cos(\omega_c t)$. But you would wind up dividing by zero, which is troublesome in theory and an absolute disaster in practice.

[1] We put "airwaves" in air quotes because you do not really need air for radio waves to propagate.

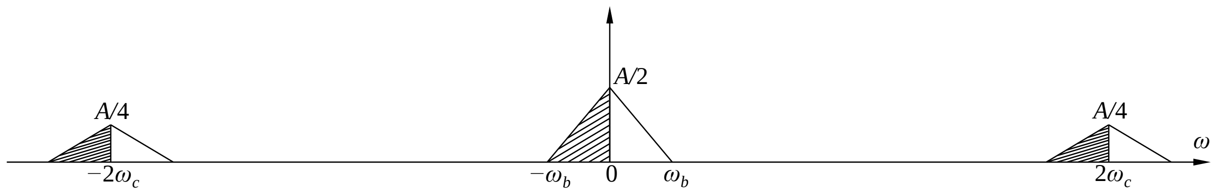


Figure 8.4: $R(j\omega)$, the Fourier transform of the mostly-demodulated signal $r(t)$.

Even dividing by things "close to" zero is problematic, whether you are using analog circuitry, computers, or magic spells.

The real first step of retrieving $x(t)$ from $m(t)$ is somewhat counterintuitive: the receiver multiplies by $\cos(\omega_c t)$ again! Let us denote this as $r(t) = m(t)\cos(\omega_c t)$, as shown in the right portion of Figure 8.2.

If we let $M(j\omega)$ play the role of $X(j\omega)$ on the right hand side of (8.14), we discover

$$R(j\omega) = \frac{1}{2}M(\omega - \omega_c) + \frac{1}{2}M(\omega + \omega_c) \tag{8.15}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{2}X(j(\omega - 2\omega_c) + \frac{1}{2}X(j\omega)\right] + \frac{1}{2}\left[\frac{1}{2}X(j\omega) + \frac{1}{2}X(j(\omega + 2\omega_c))\right]\right\} \tag{8.16}$$

$$= \frac{1}{4}X(j(\omega - 2\omega_c)) + \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega + 2\omega_c)), \tag{8.17}$$

shown in Figure 8.4. Notice we have a copy of the original innocently sitting there at DC. All we have to do is filter out the high-frequency copy at $2\omega_c$. Now you see why we reviewed brickwall filters in Section 8.1.1; we just need something that rejects frequencies above ω_{co}, where $\omega_b < \omega_{co} < (2\omega_c - \omega_b)$:

$$H(j\omega) = \begin{cases} 2 & \text{for } \omega \leq \omega_{co} \\ 0 & \text{for } \omega > \omega_{co} \end{cases}. \tag{8.18}$$

The 2 cancels the 1/2 in the second term of (8.17), so we get $X(j\omega)$ back exactly, and hence $x(t)$. (Including this "2" is somewhat pedantic; any real AM communication process will be subject to all sorts of global scaling factors, including the listener's volume control.)

We could have analyzed this process in the time domain. A trignometric identity gives us:

$$r(t) = x(t)\cos^2(\omega_c t) = \frac{x(t)}{2}[1 + \cos(2\omega_c)]. \tag{8.19}$$

By now, you should be able to quickly see that this matches the formula for $R(j\omega)$ we derived in 8.17. But it is far more illuminating to perform the entire exploration in the frequency domain.

Notice for all of this to work, we need $\omega_c > 2\omega_b$, or else the modulated copies will overlap.

8.3.3 Practical matters

The FCC requires commercial AM radio stations to limit the effective bandwidth of their broadcasts so they do not interfere with one another. These stations are are restricted to a 20 kHz broadcast bandwidth, limiting their upper audio end to 10 kHz, which makes AM mediocre for music. Also, AM is typically more hindered by interference than FM – the reasons for that, and an analysis of FM and how to demodulate FM

Figure 8.5: An example high-frequency bandlimited signal.

Figure 8.6: An example high-frequency bandlimited signal shifted to the left in the frequency domain.

signals, are beyond the scope of this class. For reasons of bandwidth and susceptibility to interferences, AM stations nowadays generally follow "talk radio" formats.

There is one additional quirk we should mention: our explanation above is not how real AM radio actually works. The trouble with the scheme we just described is that it assumes that the sinusoids of the carriers in the transmitter and the receiver are "phase locked," which is quite difficult to obtain in practice since the distance between the transmitter and the receiver is constantly changing. AM radio came into fruition a century ago; it needed to be able to work with an extremely simple receiver, even so-called "crystal radios," so a more robust modulation and demodulation scheme was needed. The solution, essentially, is to transmit the carrier along with the modulated signal. A simple demodulation process then consists of rectifying (i.e. taking the absolute value) of the received signal and lowpass filtering it to smooth out the ripples.

8.4 Baseband representations of bandlimited signals

In both analog and digital contexts, it can be difficult to design hardware that can handle high-frequency signals. In an analog design, stray capacitances between cables, copper traces on a PCB, and even the layers of an integrated circuit effectively form parasitic lowpass filters. As reviewed in Chapter 9, straightforward representation of a continuous-time signal with discrete-time samples requires a sample rate that is greater than twice the highest frequency of the underlying signal, which can place tremendous strain on analog-to-digital converters and digital signal processing hardware. Faster ADC rates are generally only obtainable with a sacrifice of bit depth, and a variety of factors give CMOS logic a practical maximum clock rate of around 4 GHz.

Suppose we wanted to investigate a signal $x(t)$ whose Fourier transform spans 9 to 11 GHz, as shown in Figure 8.5. We could say that this has a center frequency of 10 GHz and a bandwidth of 2 GHz. Radar engineers refer to such frequencies as "X-band," although the signal could also represent attempts at communication by alien civilizations from other solar systems. A naive approach would be to sample this signal at higher than 22 GHz, which is twice the highest frequency in the signal. A more sophisticated approach would be to exploit the fact that we are uninterested in frequencies up to 9 GHz. This observation leads to the idea of *quadrature demodulation* and *baseband representations*. Although we are using the term "demodulation," this technique has applications beyond communication systems; the signal being sensed does not necessarily have to result from some modulation scheme.

We must first apply an analog bandpass filter to ensure that any undesired frequencies outside the band of interest (below 9 GHz and above 11 GHz in our example) have been effectively eliminated. Then, multiplying $x(t)$ by a *complex* sinusoid carrier with a negative frequency of ω_c yielding $x_{\text{left}}(t) = x(t)\exp(-j\omega_c)$, corresponds to shifting the two-sided frequency system to the *left* by ω_c, which in our example is $\omega_c = 20\pi \times 10^9$ radians/second, as shown in Figure 8.6.

Applying an appropriate lowpass filter removes the conjugate mirror copy at -20 GHz, i.e., $-40\pi \times 10^9$

Figure 8.7: An example complex baseband signal, shown in the Fourier domain.

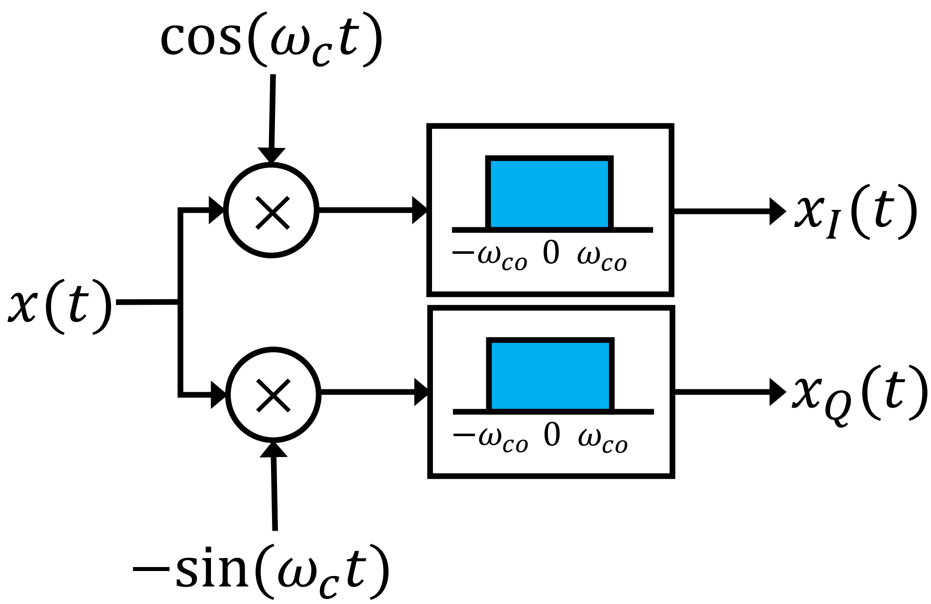


Figure 8.8: Hardware implementation of a baseband demodulation scheme.

radians/second, leaving a signal, $x_b(t)$, spanning -1 to 1 GHz in the frequency domain, as shown in Figure **??**. The signal $x_b(t)$ contains all the information embedded in the original signal $x(t)$, and retains that information if it is sampled at 2 GHz. We call $x_b(t)$ the complex baseband representation of $x(t)$. Most time-domain signals we explore in this text are real-valued, but $x_b(t)$ is complex, so extra caution is called for. In particular, $X_b(j\omega)$ does not generally possess the conjugate symmetry we are accustomed to.

At first glance, the initial multiplication by $\exp(-j\omega_c)$ looks strange, since $\exp(-j\omega_c) = \cos(-j\omega_c) + \sin(-j\omega_c) = \cos(j\omega_c) - \sin(j\omega_c)$ is complex valued, and thinking of complex numbers as Lovecraftian horrors horrors entwined with the murkiest mysteries of the universe does little to assuage that impression. But from another view, complex numbers are just shorthand for pairs of numbers that obey some useful rules. In the realm of analog circuits, "real" and "imaginary" parts can be represented as separate voltages or currents on separate wires (see Figure 8.8), and it is up to the hardware to designer to keep track of the "real part" (called the "in-phase signal") and the "imaginary part" (called the "quadrature signal").

This scheme requires duplicate hardware: two analog multipliers, two oscillators with the same frequency with phases 90 degrees apart, and two A/D converters that sample at the same time. In particular, keeping the oscillators synchronized with the appropriate phase difference is a difficult in practice for high carrier frequencies. Hence, many practical schemes employ multiple demodulation stages, where initial analog stage brings the signal down to an intermediate frequency using a single oscillator, single multiplier, and single A/D converter, and the remaining demodulation to baseband is performed in the digital domain.

8.5 Example: Fourier transform of decaying sinusoids

The modulation property has applications beyond communication systems. For instance, chemists using *nuclear magnetic resonance* and radiologists using *magnetic resonance imaging*[2] spend a lot of time thinking about *free induction decay signals*, which take the form of a decaying sinusoid, for instance,

$$x(t) = A \exp\left(-\frac{t}{T_2}\right) \sin(\omega_0 t) u(t).$$

The nonnegative constant T_2 is called the *spin-spin relaxation time.* Different kinds of body tissue have different T_2 times; for instance, by determining T_2 times, doctors can identify and locate some kinds of cancer. ω_0 is called the resonant frequency; in an MRI scheme, this tells you what part of the body the signal is coming from. A is called the *spin density*, which also depends on tissue type.

To most easily employ the Fourier transform paris we have already developed, it helps to think of the $u(t)$ factor as being part of $\exp(-t/T_2)$, not $\sin(\omega_0 t)$. We have the Fourier transform pairs

$$\exp(-t/T_2)u(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{j\omega + 1/T_2}, \tag{8.20}$$

$$\sin(\omega_0 t) \overset{\mathcal{F}}{\Longleftrightarrow} -j\pi\delta(\omega_0) + j\pi\delta(\omega + \omega_0). \tag{8.21}$$

Using the modulation property, we have

$$\begin{aligned}
X(j\omega) &= \frac{A}{2\pi}\left[\frac{-j\pi}{j(\omega - \omega_0) + 1/T_2} + \frac{j\pi}{j(\omega + \omega_0) + 1/T_2}\right] && (8.22)\\
&= \frac{A}{2}\left[\frac{-j}{j(\omega - \omega_0) + 1/T_2} + \frac{j}{j(\omega + \omega_0) + 1/T_2}\right] && (8.23)\\
&= \frac{Aj}{2}\left[\frac{-1}{(j\omega + 1/T_2) - j\omega_0} + \frac{1}{(j\omega + 1/T_2) + j\omega_0)}\right] && (8.24)\\
&= \frac{Aj[-(j\omega + 1/T_2) - j\omega_0 + (j\omega + 1/T_2) + j\omega_0]}{2[(j\omega + 1/T_2) - j\omega_0][(j\omega + 1/T_2) + j\omega_0]} && (8.25)\\
&= \frac{Aj(-2j\omega_0)}{2[(j\omega + 1/T_2)^2 + \omega_0^2]} = \frac{A\omega_0}{(j\omega + 1/T_2)^2 + \omega_0^2}. && (8.26)
\end{aligned}$$

[2]To be consistent, what we know now as MRI should be called NMRI. However, many of these developments occured during the so-called “Cold War,” and using the word “nuclear” was considered unwise from a marketing perspective. Hence, the “N” got dropped.

Chapter 9

Sampling and Periodicity

9.1 Sampling time-domain signals

Quite early in ECE2026, we introduced the notion creating a discrete-time signal by *sampling* a continuous-time signal, along with a convenient slight abuse of notation: $x[n] = x(nT_s)$, where T_s is the period between samples. $f_s = 1/T_s$ is the *sample rate*.

We introduced the idea of the Nyquist rate of $x(t)$, which is the twice the highest frequency component in $x(t)$. The Nyquist sampling theorem says that to be able to reconstruct $x(t)$ from its samples $x[n]$, you need to sample at a rate greater than the Nyquist rate.

9.1.1 A Warm-Up Question

Consider a periodic square wave with fundamental frequency f_0 (in Hertz). What is the Nyquist rate, if it exists, for this signal? In other words, is there minimum sample rate such that if we sample at higher than that sample rate, we can reconstruct $x[n]$ from its samples? If so, what is it?[1]

This is kind of a trick question. The square wave is not bandlimited—its Fourier coefficients approach zero as the harmonic number k increases, like $\mathcal{O}(1/k)$, but they never actually hit zero and stay there. So there is no Nyquist rate for a square wave!

It is not just discontinuities like those in a square wave that cause problems. A triangle wave is continuous, but its derivative has a discontinuity, and that causes problems. A triangle wave has coefficients that decreases like $\mathcal{O}(1/k^2)$ instead of $O(1/k)$, so they tend towards zero much faster than the coefficients of a square wave, but they still never exactly stay on zero.

9.1.2 Sampling: from ECE2026 to ECE3084

To keep the explanation relatively simple, Chapter 4 of *Signal Processing First* mostly addressed sampling sinusoids. You solved problems in which we gave you an input frequency and a sample rate, and asked you for the output frequency; we told you output frequency and the sample rate, and asked you what input frequencies could have produced that output; and we gave you an input frequency and an output frequency, and asked you what sample rates could have produced that output.

[1] Many years ago, one of your authors, Aaron Lanterman, put this question on a PhD written qualifying exam. At least a third of the students taking it reflexively doubled the frequency f_0 and erroneously claimed $2f_0$ to be the answer.

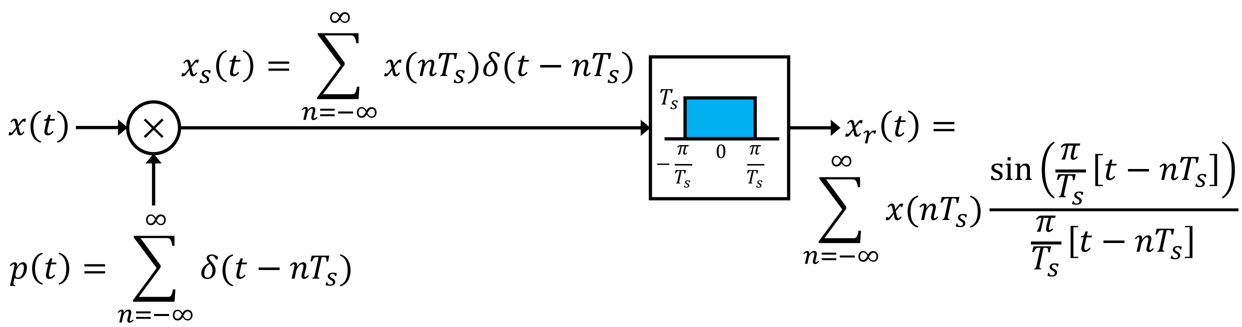


Figure 9.1: Block diagram of a mathematical model of sampling (in time) and reconstruction.

We introduced the idea of a normalized discrete-time frequency of discrete-time sinusoids, $\hat{\omega} = \omega/f_s$, or less commonly, $\hat{f} = f/f_s$.

Sampled data are ubiquitous, but the effects of sampling are often vexingly counterintuitive, so we believe it is important to address sampling early in electrical engineering and computer engineering curricula. But now that we have covered Fourier transforms, we can fill in some of the gaps (pun slightly intended) of the sampling discussion in ECE2026:

- We are no longer restricted to talking about sampling *countable* sums of sinusoids; we can mathematically describe the process of sampling any signal of practical interest in engineering, since we can represent such signals as sums of sinusoids – *uncountable sums*, if needed – via an inverse Fourier transform.
- We stated the Nyquist sampling theorem in ECE2026, but we did not fully prove it. Fourier transform theory gives us the tool we need to prove it.
- Practical sample reconstruction systems usually "fill in the gaps" between samples using straight horizontal lines ("sample and hold"), making the reconstruction look something like a Riemann sum approximation to an integral. Usually some additional analog filtering smooths the resulting "staircase." With Fourier theory, we can analyze that process more thoroughly (see Section 9.1.4), and also give an *exact*, although impractical, formula for reconstructing a signal from its samples (see Section 9.1.3).

9.1.3 A mathematical model for sampling

Consider this unit impulse train, with the Dirac deltas spaced every T_s seconds (or whatever time unit):

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s). \tag{9.1}$$

We can approximate the sampling process as multiplying a signal $x(t)$ by this impulse train $p(t)$, as shown in the left portion of Figure 9.1. The weights of the impulses in the resulting impulse train $x_s(t) = x(t)p(t)$

are the samples:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s). \tag{9.2}$$

You can imagine that an ideal continuous-to-discrete converter consists of a multiplier computing $x_s(t)$ followed by a "pull off the weights of the impulses" box that produces $x[n] = x(nT_s)$. Remember, this is a conceptual model. If you buy an ADC (analog-to-digital converter) chip from Maxim or Analog Devices, pop it open, and stick it under a microscope, you will not find any circuits in there that generate trains of Dirac delta functions and you will not find any analog multipliers. This abstract model will let us analyze sampling using Fourier theory.

Recall that we can represent periodic signals as a Fourier series summation (we are using f here instead of x so it will not be confused with the x we want to use the represent the input to our ideal ADC):

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(jk\omega_0 t\right). \tag{9.3}$$

In a previous lecture, we saw that we can take the Fourier transform of this Fourier series summation:

$$F(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - k\omega_0\right). \tag{9.4}$$

As we mentioned before, this can be a little confusing; the spectrum plots for Fourier series that we made in ECE2026 (and that many others make when talking about Fourier series) did not have the 2π. That constant is needed to get the forward and inverse Fourier transform formulas to work out.

We can easily compute the Fourier series coefficients of $p(t)$:

$$a_k = \frac{1}{T_s}\int_{-T_0/2}^{T_0/2} \delta(t)\exp(-jk\omega_s t) = \frac{1}{T_s}. \tag{9.5}$$

Hence, the Fourier transform of our pulse train $p(t)$ is

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s}\delta\left(\omega - k\omega_s\right). \tag{9.6}$$

Notice that the Fourier transform of an impulse train is another impulse train! There are other examples of such functions – for instance, sinc functions and Gaussian functions[2] – but in general, the transforms of functions do not look anything like the original functions, so this is an interesting special case.

In particular, single impulses transform into *constants*, which are about as different from an impulse as you can imagine. But our result on impulse trains is conceptually consistent. If you let $T_s \to \infty$ in $p(t)$, notice that the deltas in $P(j\omega)$ become smushed closer and closer together, so if you handwave and squint a bit, you can imagine this forming a horizontal line.

[2] We have not discussed Gaussian functions, so do not worry if have not heard of them before. You might have seen them in a class on probability.

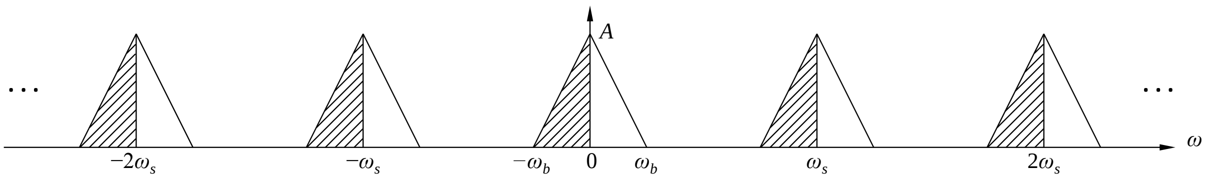


Figure 9.2: An illustration of sampling without aliasing.

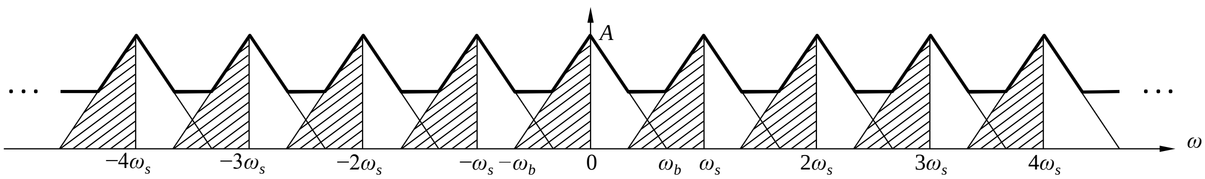


Figure 9.3: An illustration of sampling with aliasing.

Recall the modulation property from Chapter 8.2, which said that multiplication in the time domain corresponds to convolution in the frequency domain divided by 2π. Hence, the Fourier transform of $x_s(t)$ is

$$X_s(t) = \frac{1}{2\pi}\left[X(j\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s}\delta\left(\omega - k\omega_s\right)\right] \tag{9.7}$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)). \tag{9.8}$$

We see that sampling in the time domain results in periodic replication in the frequency domain. This should not be a surprise; in ECE2026, we discovered that discrete-time frequencies $\hat{\omega}$ had an ambiguity of 2π, so we limited frequency response plots of discrete-time filters to $-\pi \leq \hat{\omega} \leq \pi$.

Like in the chapter on AM communication, we will draw a triangle representing a real-valued, "generic" bandlimited signal $X(j\omega)$, with $X(j\omega) = 0$ for $\omega \geq \omega_b$, that we want to sample. We shade the left half to keep track of conjugate pairs. Again, we do not expect this Fourier transform to actually *be a triangle*; it is just a bookkeeping convention.

As illustrated in Figure 9.2, if $\omega_s \geq 2\omega_b$, the triangle copies do not overlap, so if we run $x_s(t)$ through a brickwall lowpass filter with cutoff $\omega_b \leq \omega_c \leq (\omega_s - \omega_b)$, we can keep the copy in the middle and eliminate the remaining aliases. To get the math to work out so that the reconstruction exactly equals the input, we can set the gain of the filter as T_s—but this is the sort of thing you only ever see in a textbook or on a whiteboard, since realistic systems are subject to all sorts of gain factors.

This analysis rounds out the Nyquist sampling theorem:

- It tells us why we need f_s to be greater than the Nyquist rate in general; if it is not, the triangles will overlap, and the overlapping parts will add together and we will not be able to unscramble them, as illustrated in Figure 9.3. This is not a novel observation, per se; we saw the exact same issue in ECE2026, we just drew arrows for spectral lines for particular frequency components and explored aliasing that way, instead of contemplating the full continuum of frequency components we can consider using Fourier theory.

- It tells us what the ideal reconstruction formula is. Consider the edge case of a triangle that is as wide as possible without overlapping, i.e. $\omega_{co} = \omega_b$, $\omega_s = 2\omega_b = 2\omega_{co}$, so $\omega_{co} = \omega_s/2$. From a Fourier transform pair we computed in a previous lecture, the brickwall filter with gain T_s and $\omega_{co} = \omega_s/2 = (2\pi/T_s)/2 = \pi/T_s$ has the impulse response

$$h_r(t) = \frac{T_s}{\pi t}\sin(\omega_{co}t) = \frac{1}{\frac{\pi}{T_s}t}\sin\left(\frac{\pi}{T_s}t\right).$$

  Applying the reconstruction filter in the time-domain yields the reconstruction formula

$$\begin{aligned} x_r(t) &= h_r(t) * x_s(t) && (9.9)\\ &= \frac{\sin\left(\frac{\pi}{T_s}t\right)}{\frac{\pi}{T_s}t} * \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)\right] && (9.10)\\ &= \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\left(\frac{\pi}{T_s}(t-nT_s)\right)}{\frac{\pi}{T_s}(t-nT_s)}, && (9.11) \end{aligned}$$

  as illustrated in the right portion of Figure 9.1.

  This is called the sinc interpolation formula. Notice that the zero crossing of $h_r(t)$ are at integer multiples of T_s, except for $t = 0$, where L'Hopital's rule gives us $h_r(0) = 1$. The formula tells us to reconstruct a signal by adding up sinc functions centered at the sample points with weights given by the sample values. Notice that at an integer multiple of T_s, say $t = mT_s$ (here, we are using m to avoid confusion with the n in the summation), only one sinc function contributes to $f_r(mT_s)$ because the other sinc functions are zero at that point. For spaces in between the sample points, the sinc functions conspire to "fill in the blanks."

  Notice that this "ideal" reconstruction procedure cannot be implemented in practice because of the infinite time extent and non-causality of the sinc function; however, it does provide a starting point for deriving many sophisticated computer-based interpolation and multirate DSP techniques. We will leave such matters for a senior and graduate level class in DSP.

We have seen that discrete-to-continuous converters cannot be built; some of the higher frequency copies are bound to leak through whatever practical analog reconstruction filter we construct. But we can try to get close.

There are also practical issues on the sampling side, not just the reconstruction side. Theoretically speaking, none of the signals we encounter in practice are truly bandlimited, so there is always some aliasing creeping into the original sampled signal, and in general, there is nothing we can do on the reconstruction side to fix that. However, such aliased signals may be so tiny that they fall below the effective "noise floor" (either electrical noise or limited bit resolution) of your system.

9.1.4 Practical reconstruction from samples

The preceding discussion about sampling, particularly reconstruction, was mostly of theoretical interest. It was a story fabricated to prove the Nyquist sampling theorem. Besides the question about whether you can or cannot[3] build a perfect brickwall lowpass filter, no one would design a reconstruction system that actually generated a weighted impulse train.

In practice, the typical approach is that each sample value, when converted from discrete to continuous time, is held constant until the next sample.

Instead of using a sinc function as the impulse response of the reconstruction filter, imagine using a reconstruction filter whose impulse response is a pulse of length T_s (see Figure 9.4):

$$h_0(t) = \begin{cases} 1 & \text{for } 0 \le t < T_s \\ 0 & \text{otherwise} \end{cases} . \tag{9.12}$$

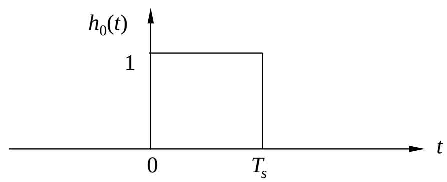


Figure 9.4: Impulse response of a zero order hold.

We are using a subscript 0 since this strategy is often called a "zero order hold." Notice that this filter is causal. It is also easy to implement because of the sampled nature of the input—a filter like this that would operate on a more generic continuous-time signal would be quite challenging to construct. Here, all the discrete-to-continuous converter needs to do is grab a value and hold it at periodic intervals; this easy to construct using a capacitor and some sort of electronic switch, implemented, for instance, in CMOS. So although it effectively *acts like* a continuous-time filter operating on a input train of Dirac delta functions, this is not a typical continuous-time filter in the sense that the word "filter" is usually used. The reconstructed signal (see Figure 9.5 for a typical example) is:

$$x_{r_0}(t) = h_0(t) * x_s(t) \tag{9.13}$$

$$= h_0(t) * \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \right] \tag{9.14}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)h_0(t - nT_s). \tag{9.15}$$

The frequency response of this zeroth-order hold filter is:

$$H_0(j\omega) = \frac{\sin(\omega T_s/2)}{\omega/2} \exp(-j\omega T_s/2). \tag{9.16}$$

The sinc part of (9.16) is the Fourier transform of a boxcar of length T_s centered at zero. The exponential tacked on at the end corresponds to a $T_s/2$ time shift. Notice that the zeros of this frequency response are at $\omega = k2\pi/T_s = k\omega_s$, where k is a nonzero integer. Hence, the zeros land at the centers of the aliased copies of the sampled signal; although this is convenient, the frequency content elsewhere still bleeds through. We

[3] You cannot.

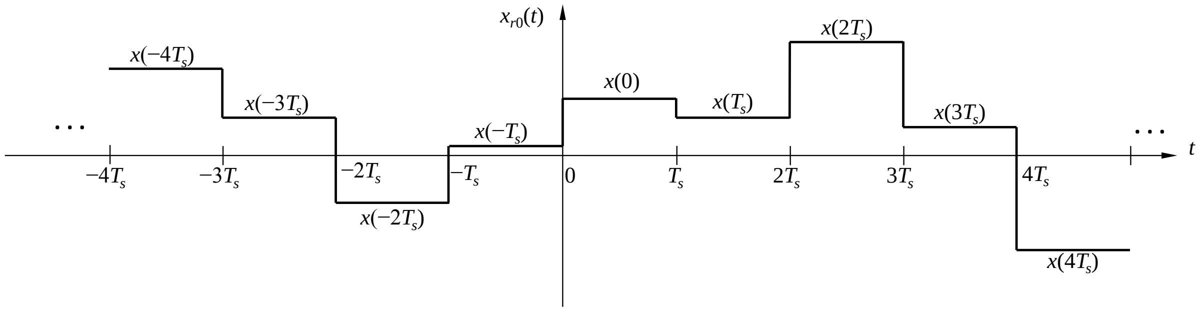


Figure 9.5: Example of reconstruction using a zero-order hold.

usually still need an analog filter to help reduce these aliases; however, since the aliases start of at a smaller amplitude, the requirements on our filter are less stringent in terms of getting reasonable but practical results. A brickwall filter, if one could somehow magically exist, might still seem ideal in terms of eliminating the aliased copies, but no matter what filter is used, notice that the spectral copy in the center that we do want to keep will have been shaped a bit by the mainlobe of the sinc function in (9.16). This discussion suggests that it can be helpful to oversample signals, i.e. sample at a rate faster than what the Nyquist sampling theorem would demand. The resulting "stairsteps" are narrower, providing a better raw approximation of the underlying signal. You can interpret this in the frequency domain by realizing that narrower pulses correspond to broader mainlobes. With significant oversampling, you can greatly relax requirements on your reconstruction filter, and filters with gentler slopes are generally easier to design and cheaper to build.

Notice that the reconstructed signal is slightly delayed relative to the underlying signal that had been sampled. This is usually not too troublesome, although delays can cause difficulties in control system design (for instance, in a large factory where control signals must travel some relatively large distance) if they are not compensated for.

9.2 Deriving the DTFT and IDTFT from the CTFT and ICTFT

Recall this model of the sampling process, presented Section 9.1.3:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s), \tag{9.17}$$

where $x[n] = x(nT_s)$ represents a sampled sequence.

We computed the Fourier transform of $x_s(t)$ in terms of $X(j\omega)$, the Fourier transform of the underlying sampled signal $x(t)$:

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)), \tag{9.18}$$

where $\omega_s = 2\pi x_s = 2\pi/T_s$.

But we could also take the Fourier transform of $x_s(t)$ more straightforwardly and express it in terms of the discrete sequence $x[n]$:

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n T_s). \tag{9.19}$$

Using the normalized frequency $\hat{\omega} = \omega / x_s = \omega T_s$, we can rewrite this as something called the Discrete-Time Fourier Transform (DTFT):

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\hat{\omega} n). \tag{9.20}$$

By convention, $X(e^{j\hat{\omega}})$ is usually plotted with the horizontal axis ranging over $-\pi \leq \hat{\omega} \leq \pi$. The DTFT $X(e^{j\hat{\omega}})$ is just a frequency-scaled version of $X_s(j\omega)$, and is periodic with period π, just as $X_s(j\omega)$ was periodic with period ω_s.

The inverse continuous-time Fourier transform of $X_s(j\omega)$ would be

$$x_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_s(j\omega) \exp(j\omega t) d\omega. \tag{9.21}$$

Remember that $x_s(t)$ is a Dirac delta train. At $t = nT_s$:

$$x_s(nT_s)\delta(t - nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_s(j\omega) \exp(j\omega n T_s) d\omega. \tag{9.22}$$

Making the substitutions $\omega = \hat{\omega}/T_s$ and $d\omega = d\hat{\omega}/T_s$, we can write the right-hand side of (9.22) in terms of the DTFT:

$$x_s(nT_s)\delta(t - nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_s(e^{j\hat{\omega}}) \exp(j\hat{\omega} n) \frac{d\hat{\omega}}{T_s}. \tag{9.23}$$

Remember that sampling in the time domain resulted in replication in the frequency domain, divided by T_s. If we just integrate over the central copy and multiply by T_s, that has the effect of applying the low-pass reconstruction filter discussed in the Section 9.1.3 and hence undoing the "Diracification" of the sampling:

$$x[n] = x(nT_s) = T_s \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X_s(e^{j\hat{\omega}}) \exp(j\hat{\omega} n) \frac{d\hat{\omega}}{T_s} \right] \tag{9.24}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_s(e^{j\hat{\omega}}) \exp(j\hat{\omega} n) d\hat{\omega}. \tag{9.25}$$

This expression is called the Inverse Discrete-Time Fourier Transform (IDTFT). Its most noticeable feature relative to its continuous-time counterpart is that the integral is of limited extent, because of the periodic nature of the discrete-time Fourier transform. (Of course, any period of 2π will suffice for the integration.)

There are more direct routes to deriving the DTFT and IDTFT pair. Our discussion here emphasizes the view that the DTFT and IDTFT are just stylized special cases cases of the CTFT and ICTFT. Nearly every CTFT property has some equivalent DTFT property. DTFT pairs and properties and their applications are thoroughly covered in senior-level courses on "digital signal processing." We wanted to paint a "big picture" of how these various transforms relate. This text focuses on the continuous-time pairs; we will let other textbooks fill in most of the blanks on the other variations.

9.3 Fourier series reimagined as frequency-domain sampling

We often use 0 as a generic subscript to indicate some constant time, like T_0, or constant frequency, like ω_0. There will be several such constants floating around in this section, so instead of using 0 as a subscript, will use letters to try to indicate what the various constants mean.

Section 7.7.1 showed that a rectangular window, symmetric around the origin, with unit height and length L, had the Fourier transform

$$X(j\omega) = \frac{2}{\omega}\sin\left(\frac{L}{2}\omega\right), \tag{9.26}$$

where $X(j0) = L$ is a special case.

In Section 6.4, we discovered that a unipolar square wave, symmetric around the origin, with unit height and a 50% duty cycle, had the Fourier series coefficients

$$a_k = \frac{1}{\pi k}\sin\left(\frac{\pi}{2}k\right), \tag{9.27}$$

where $a_0 = \frac{1}{2}$ is a special case.

It is not a coincidence that (9.26) and (9.27) look so similar. This part of the lecture will show, in a fairly generic way, how you could derive (9.27) directly from (9.26).

Let us define a new function, $x_p(t)$, which consists of $x(t)$ repeated every T_p seconds, where the p subscripts stand for "periodic." To formally define $x_p(t)$, it will be convenient to use an unit impulse train, with the Dirac deltas spaced every T_p seconds (or whatever the time unit is) apart:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_p). \tag{9.28}$$

In Section 9.1.3, we modeled sampling in time by multiplying $p(t)$ by $x(t)$ (where we wrote T_s instead of T_p). In this section, we convolve $p(t)$ by $x(t)$ to represent the periodic replication of $x(t)$: $x_p(t) = x(t) * p(t)$.

As usual, we now ask: what happens in the frequency domain?

Recollect that the Fourier transform of $p(t)$ is another impulse train:

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_p}\delta\left(\omega - k\omega_p\right), \tag{9.29}$$

where $\omega_p = 2\pi/T_p$.

Using the property that convolution in time corresponds to multiplication in frequency (*without* the 2π factor that shows up in the modulation property), we have

$$\begin{aligned} X_p(j\omega) &= X(j\omega)P(j\omega) && (9.30)\\ &= X(j\omega)\left[\sum_{k=-\infty}^{\infty} \frac{2\pi}{T_p}\,\delta\left(\omega - k\omega_p\right)\right] && (9.31)\\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_p} X(jk\omega_p)\delta\left(\omega - k\omega_p\right). && (9.32) \end{aligned}$$

That looks just like the expression we found for the Fourier transform of a periodic signal (7.9), if we set

$$a_k = \frac{X(jk\omega_p)}{T_p} = \frac{X\left(jk\frac{2\pi}{T_p}\right)}{T_p}. \tag{9.33}$$

This is extremely convenient if we want to compute the Fourier series for a periodic signal, and we can find the "core" of that signal – a *single period* the periodic waveform with a "zero extension" – in one of our Fourier transform tables.

Let us go back to our square wave example. Suppose we did not have the a_k formula for the square wave, but we were able to find $X(j\omega)$ for a single rectangular pulse in one of our Fourier transform tables. We will assume that the length L of the pulse is not any longer than the repetition interval, i.e. $L \leq T_p$, but will otherwise let L be arbitrary for now. Applying (9.33) in this case yields

$$\begin{aligned} a_k &= \frac{1}{T_p} X\left(jk\frac{2\pi}{T_p}\right) && (9.34) \\ &= \frac{1}{T_p}\frac{2}{k\frac{2\pi}{T_p}} \sin\left(\frac{L}{2}k\frac{2\pi}{T_p}\right) && (9.35) \\ &= \frac{1}{k\pi}\sin\left(Lk\frac{\pi}{T_p}\right), && (9.36) \end{aligned}$$

and for the D.C. special case, we have $a_0 = X(j0)/T_p = L/T_p$.

For a 50% duty cycle, the length is half the period, so we set $L = T_p/2$ and find

$$a_k = \frac{1}{k\pi}\sin\left(\frac{T_p}{2}k\frac{\pi}{T_p}\right) = \frac{1}{k\pi}\sin\left(k\frac{\pi}{2}\right),$$

along with the special case $a_0 = X(j0)/T_p = (T_p/2)/T_p = 1/2$. This matches the result computed in Section 6.4.

Suppose we had a square wave with a 1/3 duty cycle – i.e. the wave is 1 only 1/3 of the time and 0 the rest of the time, so $L = T_p/3$. The Fourier transform of the corresponding single pulse would be

$$a_k = \frac{1}{k\pi}\sin\left(\frac{T_p}{3}k\frac{\pi}{T_p}\right) = \frac{1}{k\pi}\sin\left(k\frac{\pi}{3}\right), \tag{9.37}$$

and for the D.C. special case, we have $a_0 = T_p/3$, which matches what we would find if we applied L'Hopital's rule to (9.37).

Notice that for a 50% duty cycle, the even harmonics are missing. For a 1/3 duty cycle, every third harmonic (3rd, 6th, 9th, etc.) is missing. This trend is consistent; a 25% duty cycle is missing the 4th, 8th, etc. harmonics.

Practical take home message: If you already have the Fourier transform pair you need, using (9.33) is a lot easier than computing the Fourier series "from scratch" using the analysis integral!

9.3.1 A quick "sanity check"

What if we "go for broke" and let $L = T_p$ in (9.36), which effectively corresponds to the constant $x(t) = 1$ in time.[4]

[4]One might argue about the exact value of $x(t)$ at multiples of T_p, but that is not important here.

Then we have

$$a_k = \frac{1}{k\pi}\sin\left(T_p k \frac{\pi}{T_p}\right) = \frac{1}{k\pi}\sin(k\pi),$$

which is zero for $k \neq 0$. For $k = 0$, L'Hopitals rule gives us

$$a_0 = \frac{\lim_{k\to 0}\pi\cos(k\pi)}{\lim_{k\to 0}\pi} = 1.$$

We could compactly write $a_k = \delta[k]$. Plugging that into our previously derived expression for the Fourier transform of a periodic function gives us $2\pi\delta(\omega)$, which Section 7.8 showed is the Fourier transform of a constant 1.

9.4 The grand beauty of the duality of sampling and periodicity

We first introduced Fourier series in ECE2026, in Chapter 3 of *Signal Processing First*; we then studied sampling in detail in Chapter 4. These probably seemed like rather different topics. But now, using the tools of Fourier transform theory, we see that the "Fourier series story" and the "sampling-in-time story" are just two manifestations of the same underlying mathematical truth: sampling in one domain corresponds to replication in the other domain.

Every "systems and signals" text we are aware introduces Fourier series first, followed by Fourier transforms. Fourier series may be a bit more intuitive than general Fourier transforms: you can "see" a waveform being built up from its individual harmonics, and periodic signals can be perceived as having a musical pitch with different harmonics contributing to its musical timbre. Most Fourier transform pairs are more opaque—a decaying sinusoid will have a Fourier transform with a bell-shaped magnitude, but it is hard to imagine how this dense continuum of sinusoids forms the underlying time-domain waveform. Hence, from an educational viewpoint, it makes sense to cover Fourier series before covering Fourier transforms, and Fourier transforms are usually motivated as a heuristic limiting case of Fourier series. However, this chapter has shown that the typical pedagogical sequence is slightly misleading; Fourier series analysis can be thought of as just a dressed-up special case of Fourier transform analysis.

In real-world applications, we usually build an apparatus that samples a time signal, and then study the effect of this sampling in the frequency domain. The way sampling in the frequency domain manifests itself usually has a more exotic flavor. Two scenarios are prevalent:

- When computing frequency responses and Fourier transforms via a traditional digital computer, we are forced to compute such functions at a finite set of frequencies. This effectively limits the extent of the time-domain signal we can study. Computational techniques tend to focus on the Fast Fourier Transform, which is an efficient algorithm for computing the Discrete Fourier Transform[5] (DFT). The DFT plays an essential role in radio astronomy, medical imaging, and radar. It is covered in ECE2026 and ECE4270, but will not play a central role in ECE3084.
- In some situations, the waveform itself inherently replicates. This is obviously the case for time-domain signals such as square waves, sawtooth waves, etc. that are easily generated with analog circuitry. But, there are also cases where we have *spatial* signals that are periodic *in space* instead of time. This is one of the most interesting aspects of X-ray crystallography; proteins are 3-D structures, but the concepts

[5] The name "Discrete Fourier Transform" is misleading and inconsistent with the most common names of some related transforms, but it has a long history and we are stuck with it.

this chapter developed in 1-D still apply. When proteins are crystallized, they form a periodic lattice. The process of X-ray diffraction allows us to measure the Fourier transform of the crystal. A century ago, the refracted radiation was recorded by a photographic plate, and these plates illustrated a pattern of dots. These dots are samples of the Fourier transform of a signal protein. A tremendous challenge arises in that we can measure the amplitude of the Fourier transform, but not the phase. This leads to the topic of "phase retrieval," which is beyond the scope of this text.

The example of X-ray crystallography brushes against an important aspect of Fourier transforms that is often not emphasized in undergraduate-level courses and textbooks on "signals and systems." Such courses are taught by, and such texts are written by, professors who typically specialize in controls, communication, or signal processing, and they typically focus on signals *in time*. Fourier theory is usually presented as a technique for understanding and analyzing continuous-time signals and systems, particularly LTI systems. But, if you study electromagnetics (particularly the field of antennas) or optics (for instance, Georgia Tech has a class called ECE6501: Fourier Optics and Holography) at the senior or graduate level, you will find two and three dimensional Fourier transforms playing a direct role. In fact, many systems in antennas and optics can be approximated as Fourier transform operations. These systems are linear, but not spatially-invariant. The distinction here cannot be overemphasized. It is not just that some optical systems can be analyzed with Fourier transforms—some of them *are Fourier transforms!*

For now, we will leave such discussions to other courses.

Fourier theory is deep and broad, and its applications are endless. We have barely scratched the surface. For further explorations, Georgia Tech as a graduate class called ECE6500: Fourier Techniques and Signal Analysis, and even more rigorous treatments can be found in math departments.

Part III: Complex Transform Domains

Chapter 10

Laplace Transforms

From this point forward, the course material may seem to get easier, in that we will restrict *most* of our discussion to systems that can be described by linear differential equations with constant coefficients. Laplace transforms are particularly convenient for analyzing such systems.

10.1 Introducing the Laplace transform

10.1.1 Beyond Fourier

If an LTI system is given by

$$\dot{y} = -ay + bx, \tag{10.1}$$

where y is the output of the system and x the input, we saw earlier that, if $y(0) = 0$ and $a > 0$, the impulse response was given by

$$h(t) = be^{-at}u(t), \tag{10.2}$$

and that the output could be obtained through the convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = h(t) * x(t). \tag{10.3}$$

Moreover, due to the awesome fact that convolution in time corresponds to multiplication in frequency, the Fourier transform of the output is given by

$$Y(j\omega) = H(j\omega)X(j\omega). \tag{10.4}$$

This is all very well, but what if $y(0) \neq 0$? Or even worse, what if $a < 0$? In this case, the uncontrolled system ($u = 0$) actually "blows up" since

$$y(t) = e^{-at}y(0) \to \infty \text{ if } y(0) \neq 0,\ a < 0. \tag{10.5}$$

We clearly need to be able to understand this type of situation as well – both from an analysis and from a design point-of-view – but the problem is that the Fourier transform of an increasing exponential does not exist since

$$\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}dt = \infty \quad \text{if } a < 0. \tag{10.6}$$

What is at play here is that the Fourier transform can be thought of as a (normalized) projection of a signal onto another, periodic, signal $e^{j\omega t}$. But to be able to capture a richer class of signals, e.g., signals that go off to infinity as $t \to \infty$, a different set of signals are needed. As such, another possible choice of signals onto which we would like to project our signals, that still look and feel very much like the Fourier transform, could be to use e^{st}, where $s = \sigma + j\omega \in \mathcal{C}$. This allows us to have exponentially decaying ($\sigma < 0$) or growing ($\sigma > 0$) sinusoids, and if we simply replace $j\omega$ with s in the Fourier transform of the signal $x(t)$, we get

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

This is called the *bilateral* or *two-sided* Laplace transform. Another form of this transform, called the *unilateral* or *one-sided* Laplace transform, is particularly elegant for studying causal systems subject to inputs for $t \geq 0$, and characterized by a set of initial conditions at $t = 0$. (Technically speaking, we will refer to initial conditions for a time infinitesimally before $t = 0$, which we will denote as $t = 0^-$.) The unilateral Laplace transform is defined as

$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt. \tag{10.7}$$

The superscript minus sign on the zero in the lower limit indicates that the $t = 0$ point needs to be included in the interval.[1] For "ordinary" functions, this distinction is not needed, but we may need it here of $x(t)$ contains a singularly such as Dirac delta function at $t = 0$. Normally, we try to avoid such seemingly nitpicky details, and most textbooks treat the $t = 0$ rather cavalierly, but without some caution it can become painfully easy to write down contradictory statements when employing one-sided Laplace transforms. Note that even though the interpretation of the Laplace transform is not as intuitively direct as the Fourier transform, its usefulness and strength is derived from its versatility as a tool for analyzing and describing systems, particularly systems described by linear differential equations with constant coefficients.

The bilateral Laplace transform is rarely used by practicing engineers, and leads to complications with dealing with "regions of convergence," as discussed later, that do not crop up when using the unilateral Laplace transform. Hence, we will focus almost entirely on employing the unilateral Laplace transform. Unless we specifically say "bilateral," all mentions of Laplace transform in this text refer to the unilateral form of (10.7).

The system outputs $y(t)$ we derive throughout this chapter should be thought of as only being valid for $y(t)$ for $t \geq 0$; we remain agnostic as to what the output might have been before $t = 0$.

10.1.2 Examples

To get started, let's consider a few examples of the Laplace transform:

$$x(t) = u(t) \;\Rightarrow\; X(s) = \mathcal{L}[u(t)] = \int_{0^-}^{\infty} e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_{t=0^-}^{t=\infty} = \frac{1}{s}(1 - \text{``}e^{-s\infty}\text{''}). \tag{10.8}$$

Note that the expression within the quotation marks is somewhat strange and the reason for this is that we actually do not know exactly what this is, since, if $s = \sigma + j\omega$,

$$\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty} e^{-\sigma t}(\cos(\omega t) + j\sin(\omega t)), \tag{10.9}$$

[1] Some books use the notation $\mathcal{L}_-$ instead of $\mathcal{L}$, and also introduce an alternate unilateral transform $\mathcal{L}_+[x(t)] = \int_{0^+}^{\infty} x(t)e^{-st}dt$ that explicitly excludes the $t = 0$ point. Although this is controversial, we believe that the $\mathcal{L}_+$ formulation offers little value compared with $\mathcal{L}_-$, so we simply use $\mathcal{L}_-$ throughout and leave out the subscript.

which is equal to zero only if $\sigma > 0$. (If $\sigma < 0$ it is in fact infinity and $\sigma = 0$ means that the limit does not exist.) As such, we need to be a bit careful when taking the Laplace transforms and actually ensure that the transform is indeed defined. For $\sigma > 0$ we thus have that $\mathcal{L}[u(t)] = 1/s$ and we will denote this by

$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \text{Re}(s) > 0. \tag{10.10}$$

Note that most of the time it actually does not matter that s is restricted to its so-called region of convergence for the existence of the transform and we will, in practice, almost always ignore this restriction. We are able to get away with this since we are studying the unilateral Laplace transform.

Now, let's consider the exponential $x(t) = e^{-at}u(t)$:

$$X(s) = \mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = \frac{1}{a+s}(1 - \text{“}e^{-(s+a)\infty}\text{”}), \tag{10.11}$$

where we again have to restrict s to have a real part that is greater than $-a$ for the transform to exist, i.e.,

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}, \quad \text{Re}(s) > -a. \tag{10.12}$$

10.2 Key properties of the Laplace transform

Just like the Fourier transform came with a collection of useful properties, the Laplace transform has its own useful properties. We here discuss some of the key such properties.

10.2.1 Linearity

As the Laplace transform is obtained through and integral, which itself is a linear operation, linearity is inherited by the transform, since

$$\begin{aligned}\mathcal{L}[p_1 f_1(t) + p_2 f_2(t)] &= \int_{0^-}^{\infty} (p_1 f_1(t) + p_2 f_2(t))e^{-st}dt \\ &= p_1 \int_{0^-}^{\infty} f_1(t)e^{-st}dt + \alpha_2 \int_{0^-}^{\infty} f_2(t)e^{-st}dt = p_1\mathcal{L}[f_1(t)] + p_2\mathcal{L}[f_2(t)].\end{aligned}$$

As an example, consider

$$x(t) = \sin(\omega_0 t)u(t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})u(t). \tag{10.13}$$

From linearity and the transform of the exponential, it follows that

$$X(s) = \frac{1}{2j}\left(\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right) = \frac{\frac{s+j\omega_0 - s + j\omega_0}{(s-j\omega_0)(s+j\omega_0)}}{2j} = \frac{2j\omega_0}{2j(s^2 - (j\omega_0)^2)} = \frac{\omega_0}{s^2 + \omega_0^2}. \tag{10.14}$$

In exactly the same manner, we can compute the Laplace transform of $\cos(\omega_0 t)$, which is in fact given by

$$\mathcal{L}[\cos(\omega_0 t)] = \frac{s}{s^2 + \omega_0^2}. \tag{10.15}$$

10.2.2 Taking derivatives

One of the main reasons Laplace transforms are so useful when studying systems is that they play particularly well with differentiation operators. In other words, given that $g(t) = x'(t)$, we would like to express $G(s)$ in terms of $X(s)$. But,

$$G(s) = \mathcal{L}[g(t)] = \int_{0^-}^{\infty} x'(t)e^{-st}dt, \tag{10.16}$$

which, using integration by parts, becomes

$$G(s) = x(t)e^{-st}\Big|_{t=0^-}^{t=\infty} - \int_{0^-}^{\infty} x(t)(-s)e^{-st}dt = 0 - x(0^-) + sX(s). \tag{10.17}$$

So, to summarize, we have that

$$\mathcal{L}[x'(t)] = s\mathcal{L}[x(t)] - x(0^-), \tag{10.18}$$

where $x(0)$ is the initial condition for $x(\cdot)$.

As an example, consider the differential equation

$$\dot{y} = -ay, \quad y(0^-) = 1. \tag{10.19}$$

Taking the Laplace transform of both sides of this equation yields

$$sY(s) - y(0^-) = -aY(s) \ \Rightarrow \ Y(s)(s+a) = y(0^-) \ \Rightarrow \ Y(s) = \frac{1}{s+a}. \tag{10.20}$$

But, we have already seen that

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}, \tag{10.21}$$

and, as such, we can conclude that

$$y(t) = e^{-at}u(t). \tag{10.22}$$

What we actually did was solve a differential equation using the fact that the Laplace transform works so well with time differentiation.

As another example, consider

$$\mathcal{L}[x''(t)] = \mathcal{L}\left[\frac{d}{dt}x'(t)\right] = s\mathcal{L}[x'(t)] - f'(0^-) = s^2X(s) - sx(0^-) - x'(0^-). \tag{10.23}$$

10.2.3 Integration

In exactly the same way as for differentiation, we have that

$$\mathcal{L}\left[\int_0^t x(\tau)d\tau\right] = \frac{1}{s}X(s). \tag{10.24}$$

As an example of this, let $x(t) = tu(t)$, and we note that

$$tu(t) = \int_0^t d\tau = \int_0^t u(\tau)d\tau. \tag{10.25}$$

Hence

$$\mathcal{L}[tu(t)] = \frac{1}{s}\mathcal{L}[u(t)] = \frac{1}{s^2}. \tag{10.26}$$

What if $x(t) = t^2u(t)$? Well, we can use this method again, since we know that

$$\int_0^t \tau d\tau = \left[\frac{\tau^2}{2}\right]_{\tau=0}^t = \frac{t^2}{2}u(t). \tag{10.27}$$

As such,

$$\mathcal{L}[t^2u(t)] = 2\mathcal{L}\left[\int_0^t \tau d\tau\right] = \frac{2}{s}\mathcal{L}[tu(t)] = \frac{2}{s^3}. \tag{10.28}$$

10.2.4 Time delays

Just as for the Fourier transform, time-delays translates into multiplication of the transform by a particular exponential. What we would like to do is compute the Laplace transform of $x(t-T)$. But, since we have to worry about initial conditions when dealing with Laplace transforms, we need to specify $x(t)$ not only for $t \geq 0$ but also for $t \geq -T$. If we assume that $x(t) = 0,\ \forall t \in [-T, 0)$, then we can directly compute the Laplace transform

$$\mathcal{L}[x(t-T)] = \int_0^\infty x(t-T)e^{-st}dt = \int_T^\infty x(t-T)e^{-st}dt = e^{-sT}\int_0^\infty x(\tau)e^{-s\tau}d\tau = e^{-sT}X(s), \tag{10.29}$$

which, as it turns out, looks just like the time-delay property for Fourier transforms expect that we have replaced the imaginary $j\omega$ with the complex s.

For example, let $x(t) = u(t-1)$ Then

$$\mathcal{L}[x(t)] = e^{-s}\mathcal{L}[u(t)] = \frac{e^{-s}}{s}. \tag{10.30}$$

If instead we have a rectangular pulse $x(t) = u(t) - u(t-1)$, we have

$$\mathcal{L}[x(t)] = \mathcal{L}[u(t) - u(t-1)] = \frac{1 - e^{-s}}{s}. \tag{10.31}$$

As another example, consider the impulse train

$$x(t) = \sum_{k=0}^\infty \delta(t-k). \tag{10.32}$$

First, we need to compute

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = \left[e^{-st}\right]_{t=0}^\infty = 1, \tag{10.33}$$

from which we have

$$\mathcal{L}[x(t)] = \sum_{k=0}^\infty 1e^{-sk} = \sum_{k=0}^\infty \left(e^{-s}\right)^k = \frac{1}{1-e^{-s}}. \tag{10.34}$$

10.3 The initial and final value theorems

Another key aspect of the Laplace transform of a signal $x(t)$ is that it allows to be able to describe what will happen to $x(t)$ asymptotically, i.e., as $t \to \infty$. This is rather important, for example when one is designing controllers that are supposed to track reference values. For example, if one where to build a cruise-controller that makes a car drive at 60 mph then it would be useful to be able to show that the speed (after a while) is indeed 60 mph and not 75 mph, which boils down to showing to what happens to the speed of the car as t becomes large.

To start with, we recall that

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = sX(s) - x(0). \tag{10.35}$$

But, we also know (from the fundamental theorem of calculus) that

$$\int_0^\infty f'(t)dt = x(\infty) - x(0), \tag{10.36}$$

if $x(\infty)$ exists (and where we have to accept a slight abuse of notation.)

Moreover, as

$$\lim_{s\to 0} \int_0^\infty f'(t)e^{-st}dt = \int_0^\infty f'(t)dt \tag{10.37}$$

we have that

$$\lim_{s\to 0}\big(sX(s) - x(0)\big) = x(\infty) - x(0), \tag{10.38}$$

from which we conclude that if $x(\infty)$ exists, the final value theorem tells us that

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s). \tag{10.39}$$

The final value theorem also has an initial-value counterpart (derived in a similar way), that states that

$$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty\times 1} sX(s). \tag{10.40}$$

Note that s approaches infinity along the positive real axis (designated by $\infty \times 1$).

10.3.1 Examples

Suppose that $x(t)$ has the Laplace transform

$$X(s) = \frac{-3s^2+2}{s^3+s^2+3s+s}. \tag{10.41}$$

Then (assuming the limit exists),

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{-3s^3+2s}{s^3+s^2+3s+2} = 0. \tag{10.42}$$

Similarly, $x(t)$'s value at time $t = 0$ is given by

$$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{-3s^3+2s}{s^3+s^2+3s+2} = -3. \tag{10.43}$$

10.4 Partial fraction expansions (PFEs)

In applying Laplace transforms to signal and system theory, we often have a need to invert Laplace transforms that are ratios of polynomials. In practice, we often need to invert the Laplace transform of the output of a system, so this section will generally notate Laplace transforms of interest as $Y(s)$. However, the methods described in this section same method could be applied to inverting Laplace representations of inputs, usually denoted $X(s)$, or any other rational function.

10.4.1 First PFE example

Suppose we need to invert following Laplace transform:

$$Y(s) = \frac{3}{s^2 + 7s + 1} \tag{10.44}$$

We can factor the denominator and write a *partial fraction expansion* (PFE):

$$Y(s) = \frac{3}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5}. \tag{10.45}$$

Partial fraction expansions allows us to express Laplace transforms as a sum of individual terms, each of which can be painlessly inverted by looking up the appropriate form on a Laplace transform table. The tricky part is finding the PFE coefficients, in this case A and B. We will describe two of the most common approaches: the "fraction-clearing" method and the "residue" method. Later chapters emphasize the residue method, but the residue method is best motivated by first understanding the fraction-clearing method.

Fraction clearing method: We can multiply each side of the equation specified by the rightmost equality in (10.45) by $(s + 2)(s + 5)$, yielding

$$3 = A(s + 5) + B(s + 2). \tag{10.46}$$

We can equate the constant on each side of (10.46) and the terms containing s on each side of (10.46):

$$\begin{aligned} s^1 &: \quad 0 = A + B \;\Rightarrow\; B = -A \;\Rightarrow\; B = -1 \\ s^0 &: \quad 3 = 5A + 2B \;\Rightarrow\; 3 = 5A - 2A = 3A \;\Rightarrow\; A = 1 \end{aligned} \tag{10.47}$$

Residue method: Instead of completely clearing the fractions, we could partially clear them by multiply both sides of (10.50) by $s + 2$, yielding a term of just A on the right hand side:

$$\frac{3}{s+5} = A + \frac{B(s+2)}{s+5}. \tag{10.48}$$

Setting $s = -2$ eliminates the B term, and we are left with

$$A = \frac{3}{-2+5} = \frac{3}{3} = 1. \tag{10.49}$$

Similarly, we could partially clear the fractions by multiplying both sides of (10.50) by $s + 5$, yielding a term of just B on the right hand side:

$$\frac{3}{s+2} = \frac{A(s+5)}{s+2} + B. \tag{10.50}$$

Setting $s = -5$ eliminates the A term, and we are left with

$$B = \frac{3}{-5+2} = \frac{3}{-3} = -1. \quad (10.51)$$

This motivates an approach to finding PFE coefficients called the *residue method*. In general, for the case of the PFE of a ratio of polynomials with N distinct roots in the denominator, the coefficients of an expansion of the form

$$Y(s) = \frac{P(s)}{(s-p_1)(s-p_2)\cdots(s-p_N)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_N}{s-p_N} \quad (10.52)$$

may be found with the formula

$$A_i = [(s-p_i)Y(s)]|_{s=p_i}. \quad (10.53)$$

The subscripts in (10.53) let us write a general recipe for an arbitrary number of distinct roots. When working problems, we recommend using different letters, since keeping track of A, B, and C seems to be less error prone than keeping track of A_1, A_2, and A_3.

The residue method is particularly compelling when only one or a few of the coefficients of the PFE are of interest, since the complete clear-the-fraction method essentially requires all of them to be found.

10.4.2 An example with distinct real and imaginary roots

Consider this PFE:

$$Y(s) = \frac{4}{s^3+3s^2+9s+27} = \frac{4}{(s^2+9)(s+3)} = \frac{4}{(s-j3)(s+j3)(s+3)} \quad (10.54)$$

$$= \frac{A}{s-j3} + \frac{B}{s+j3} + \frac{C}{s+3}. \quad (10.55)$$

Using the residue method, we find:

$$C = \left.\frac{4}{s^2+9}\right|_{s=-3} = \frac{4}{18} = \frac{2}{9}. \quad (10.56)$$

$$A = \left.\frac{4}{(s+j3)(s+3)}\right|_{s=j3} = \frac{4}{j6(3+j3)} = \frac{4}{-18+j18} = \frac{2}{9(-1+j)}. \quad (10.57)$$

To write A in a more typical rectangular form, we can multiply the numerator and denominator by the complex conjugate of $(-1+j)$:

$$A = \frac{2}{9(-1+j)} \times \frac{-1-j}{-1-j} = \frac{2(-1-j)}{9(1+1)} = \frac{-1-j}{9} = \frac{\sqrt{2}}{9}\exp(-j3\pi/4). \quad (10.58)$$

By conjugate symmetry,

$$B = A^* = \frac{-1+j}{9} = \frac{\sqrt{2}}{9}\exp(j3\pi/4) \quad (10.59)$$

Plugging in these coefficients yields

$$Y(s) = \frac{(\sqrt{2}/9)\exp(-j3\pi/4)}{s-j3} + \frac{(\sqrt{2}/9)\exp(j3\pi/4)}{s+j3} + \frac{2/9}{s+3}, \quad (10.60)$$

which has the inverse Laplace transform

$$y(t) = \frac{\sqrt{2}}{9}\exp(-j3\pi/4)\exp(j3t) + \frac{\sqrt{2}}{9}\exp(j3\pi/4)\exp(-j3t) + \frac{2}{9}\exp(-3t) \quad (10.61)$$

$$= \frac{2\sqrt{2}}{9}\cos\left(3t - \frac{3\pi}{4}\right) + \frac{2}{9}\exp(-3t), \quad \text{for } t \geq 0. \quad (10.62)$$

10.4.3 An example with complex roots

Here, we will consider two approaches for inverting

$$Y(s) = \frac{10}{s^2 + 4s + 29}. \quad (10.63)$$

First, we will try a brute-force PFE:

$$Y(s) = \frac{10}{(s + 2 - j5)(s + 2 + j5)} = \frac{A}{s + 2 - j5} + \frac{B}{s + 2 + j5}. \quad (10.64)$$

The residue method yields

$$A = \left.\frac{10}{s + 2 + j5}\right|_{s=-2+j5} = \frac{10}{-2 + j5 + 2 + j5} = \frac{10}{j10} = -j. \quad (10.65)$$

By conjugate symmetry, $B = A^\star = j$. Inserting the coefficients into (10.64) yields

$$Y(s) = \frac{10}{(s + 2 - j5)(s + 2 + j5)} = \frac{-j}{s + 2 - j5} + \frac{j}{s + 2 + j5}, \quad (10.66)$$

which has the inverse

$$y(t) = -je^{(-2+j5)t} + je^{(-2-j5)t} = e^{-2t}(-je^{j5t} + je^{-j5t}) = 2e^{-2t}\sin(5t), \quad \text{for } t \geq 0. \quad (10.67)$$

Second, we can try avoiding the PFE computation by completing the square in the denominator:

$$Y(s) = \frac{10}{s^2 + 4s + 4 + 25} = \frac{10}{(s + 2)^2 + 5^2} = 2\frac{5}{(s + 2)^2 + 5^2}. \quad (10.68)$$

From the table, $y(t) = 2e^{-2t}\sin(5t)$ for $t \geq 0$, which matches (10.67).

10.4.4 Residue method with repeated roots

Now consider a case where the denominator has both distinct roots and a repeated root. The distinct roots can be accommodated as before, but additional terms are needed to handle the repeated root:

$$Y(s) = \frac{P(s)}{(s - p_1)^r(s - p_{r+1})\cdots(s - p_N)} = \frac{A_1}{s - p_1} + \frac{A_2}{(s - p_1)^2} + \cdots + \frac{A_r}{(s - p_1)^r} + \frac{A_{r+1}}{s - p_{r+1}}\cdots + \frac{A_N}{s - p_N}. \quad (10.69)$$

The coefficients for the distinct terms still be found may be found with the using 10.53. The coefficient of highest-order term can be found the same way:

$$A_r = [(s-p_1)^r Y(s)]|_{s=p_1}. \tag{10.70}$$

The repeated terms present additional complications. One approach, which might be called a "derivative technique," discovers $A_1, A_2, \ldots, A_r - 1$ using the formula

$$A_{r-k} = \frac{1}{k!}\left[\frac{d^k}{ds^k}\{(s-p_1)^r Y(s)\}\right]\bigg|_{s=p_1}. \tag{10.71}$$

For instance, a thrice-repeated root requires the computations [MORE TO COME]

An example with only a repeated root

Consider the PFE

$$Y(s) = \frac{s^2+1}{(s+3)^3} = \frac{A_1}{s+3} + + \frac{A_2}{(s+3)^2} + \frac{A_3}{(s+3)^3}. \tag{10.72}$$

The derivative technique yields

$$A_3 = (s^2+1)|_{s=-3} = 10, \tag{10.73}$$

$$A_2 = \left[\frac{d}{ds}\{s^2+1\}\right]\bigg|_{s=-3} = 2s|_{s=-3} = -6, \tag{10.74}$$

$$A_1 = \frac{1}{2}\left[\frac{d^2}{ds^2}\{2\}\right]\bigg|_{s=-3} = 1. \tag{10.75}$$

Taking the inverse Laplace transform of

$$Y(s) = \frac{s^2+1}{(s+3)^3} = \frac{1}{s+3} + - \frac{6}{(s+3)^2} + \frac{10}{(s+3)^3}. \tag{10.76}$$

results in

$$y(t) = e^{-3t} - 6te^{-3t} + 5t^2 e^{-3t} \quad \text{for } t \geq 0. \tag{10.77}$$

An example with a repeated root and a distinct root

Consider the PFE

$$Y(s) = \frac{1}{(s+2)^2(s+1)} = \frac{A_1}{s+2} + \frac{A_2}{(s+2)^2} + \frac{B}{s+1}. \tag{10.78}$$

Notice we use the symbol B instead of A_3; we find that in the case of repeated roots, it is helpful to use different letters for different roots, and subscripts to indicate powers. The derivative technique yields

$$B = \frac{1}{(s+2)^2}\bigg|_{s=-1} = 1, \tag{10.79}$$

$$A_2 = \frac{1}{s+1}\bigg|_{s=-2} = -1, \tag{10.80}$$

$$A_1 = \left[\frac{d}{ds}\left\{\frac{1}{s+1}\right\}\right]\bigg|_{s=-2} = \frac{-1}{(s+1)^2}\bigg|_{s=-2} = -1 \tag{10.81}$$

The derivative technique can become cumbersome when the resulting expressions become more complicated (as in (10.81)) instead of simpler (as in (10.75)) with each derivative operation, particularly if you need to engage with the rule for taking the derivative of quotients. In such cases, an alternative approach is to substitute in the known coefficients along with some convenient value for s to find the remaining coefficient. Obviously, chosen s can not be one of the roots, otherwise one or more of the term will throw a "divide by zero" exception.[2] In the case of (10.81), let's try substituting $B = 1$, $A_2 = -1$, and $s = 0$ into (10.83):

$$Y(0) = \frac{1}{4} = \frac{A_1}{2} + \frac{A_2}{4} + \frac{B}{1} = \frac{A_1}{2} - \frac{1}{4} + 1 \tag{10.82}$$

$$A_1 = 2\left(\frac{1}{4} + \frac{1}{4} - 1\right) = -1 \tag{10.83}$$

While $s = 0$ was convenient, there was nothing particularly magical about it; any $s \neq -1, -2$ would suffice to generate an equation that could be solved for A_1.

Taking the inverse Laplace transform of

$$Y(s) = \frac{1}{(s+2)^2(s+1)} = -\frac{1}{s+2} - \frac{1}{(s+2)^2} + \frac{1}{s+1}. \tag{10.84}$$

results in

$$y(t) = -e^{-2t} - te^{-3t} + e^{-t} = e^{-t} - (t+1)e^{-2t} \quad \text{for } t \geq 0. \tag{10.85}$$

10.5 PFEs of Improper Fractions

In all of the PFE examples we have done so far, the degree of the denominator has exceeded the degree of the numerator. If this is not the case, we must first use polynomial division to write the Laplace transform as a sum of a quotient[3] polynomial and a strictly proper ratio:

$$Y(s) = \frac{P(s)}{Q(s)} = \mathfrak{Q}(s) + \frac{R(s)}{Q(s)}. \tag{10.86}$$

We can find the quotient $\mathfrak{Q}(s)$ and remainder $R(s)$ via polynomial long division. For example, suppose we want to find the inverse Laplace transform of

$$Y(s) = \frac{s^3 + 6s^2 + 12s + 66}{(s+4)(s^2+9)} = \frac{s^3 + 6s^2 + 12s + 66}{s^3 + 4s^2 + 9s + 36}. \tag{10.87}$$

Performing the long division

$$\begin{array}{r|r} & 1 \\ \hline s^3 + 4s^2 + 9s + 36 & s^3 + 6s^2 + 12s + 66 \\ & -s^3 - 4s^2 - 9s - 36 \\ \hline & 2s^2 + 3s + 30 \end{array} \tag{10.88}$$

[2]We apologize to the reader. The author who wrote this sentence has recently been spending too much time programming.

[3]We use a Q in "Fraktur" font for the quotient since we already used Q for the denominator polynominal, but still really, really, really wanted to somehow use the letter "Q" for quotient.

yields $Q(s) = 1$ and $R(s) = 2s^2 + 3s + 30$, so we can write

$$Y(s) = 1 + \frac{2s^2 + 3s + 30}{s^3 + 4s^2 + 9s + 36} = 1 + \frac{2s^2 + 3s + 30}{(s+4)(s^2+9)}. \tag{10.89}$$

The last expression in (10.89), with the factored denominator, is most convenient for proceeding with performing the PFE of the $R(s)/A(s)$ term:

$$Y(s) = 1 + \frac{c_1}{s - 3j} + \frac{c_1^*}{s + 3j} + \frac{c_2}{s+4} \tag{10.90}$$

Using the residue method, we find

$$c_1 = \left.\frac{2s^2 + 3s + 30}{(s+4)(s+3j)}\right|_{s=3j} = \frac{2(-9) + 3(3j) + 30}{(3j+4)(6j)} = \frac{12 + 9j}{6(-3+4j)} = \frac{4+3j}{2(-3+4j)} \cdot \frac{-3-4j}{-3-4j} \tag{10.91}$$

$$= \frac{-12 - 9j - 16j + 12}{2(9+16)} = \frac{-25j}{2(25)} = \frac{1}{2}j \tag{10.92}$$

and

$$c_2 = \left.\frac{2s^2 + 3s + 30}{s^2 + 9}\right|_{s-4} = \frac{32 - 12 + 30}{16 + 9} = \frac{50}{25} = 2. \tag{10.93}$$

Inverting

$$Y(s) = 1 - \frac{j/2}{s - 3j} + \frac{j/2}{s + 3j} + \frac{2}{s+4} \tag{10.94}$$

yields

$$y(t) = \delta(t) + \cos(t - \pi/2) + 2e^{-4t} = \delta(t) + \sin(t) + 2e^{-4t} \quad \text{for } t \geq 0. \tag{10.95}$$

10.6 Laplace and differential equations

One of the main reasons to introduce the Laplace transform is that it works well together with systems that are not initially at rest; i.e., it allows for a richer treatment of linear, ordinary differential equations.

10.6.1 First-order system example

As an example, consider

$$\dot{y} = -y + 1, \;\; y(0^-) = 0. \tag{10.96}$$

As $\mathcal{L}[u(t)] = 1/s$, and we really do not care what happens before $t = 0$, we can replace the 1 by a $u(t)$. As such, we get

$$sY(s) = -Y(s) + \frac{1}{s} \;\Rightarrow\; (s+1)Y(s) = \frac{1}{s} \;\Rightarrow\; Y(s) = \frac{1}{s(s+1)}. \tag{10.97}$$

Here, we will use the "fraction clearing method" to compute the partial fraction expansion:

$$Y(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}. \tag{10.98}$$

We could write:

$$\frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} = \frac{s(A+B)+A}{s(s+1)} = \frac{1}{s(s+1)}, \tag{10.99}$$

and multiply the last two expressions by s(s+1). By identifying the coefficients above, we get

$$\begin{array}{ll} s^1: & A+B=0 \;\Rightarrow\; B=-A \\ s^0: & A=1 \end{array} \tag{10.100}$$

As such, we have

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} \;\Rightarrow\; y(t) = [1-e^{-t}]u(t). \tag{10.101}$$

This method is quite general and it allows us to solve differential equations in a straight-forward manner.

10.6.2 Another first-order system example

As another example, consider a serial RC circuit, driven by a fixed voltage v. Kirchoff's Laws tell us that the voltage over the resistor satisfies $V_R = Ri$, where i is the current in the circuit. Moreover, the voltage over the capacitor satisfies $\dot{V}_c = i/C$. If we let $y(t) = V_c(t)$, then we have that

$$v = Ri + y. \tag{10.102}$$

But, $i = C\dot{V}_c = C\dot{y}$ gives the differential equation $v = RC\dot{y} + y$, or

$$\dot{y} + \frac{1}{RC}y = \frac{v}{RC}. \tag{10.103}$$

Now, assume that $y(0) = 0$ and $v = u(t)$. Taking the Laplace transform yields

$$sY(s) + \frac{1}{RC}Y(s) = \frac{1}{RCs} \;\Rightarrow\; Y(s) = \frac{1}{RC}\left(\frac{1}{s(s+\frac{1}{RC})}\right). \tag{10.104}$$

The PFE becomes

$$\frac{1}{s(s+1/RC)} = \frac{A}{s} + \frac{B}{s+1/RC} = \frac{As + A/RC + Bs}{s(s+1/RC)}. \tag{10.105}$$

Lining up the coefficients gives

$$\begin{array}{ll} s^0: & A/RC = 1 \;\Rightarrow\; A = RC \\ s^1: & B = -A \;\Rightarrow\; B = -RC. \end{array} \tag{10.106}$$

As such,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{RC}\left(\frac{RC}{s} - \frac{RC}{s+1/RC}\right)\right] = \frac{1}{RC}\left(RC - RCe^{-t/RC}\right)u(t) = u(t) - e^{-\frac{1}{RC}}u(t). \tag{10.107}$$

Note that in both of these examples, we only had one time-derivative (one dot over y), which means that we have a so-called first-order system. The same method for solving differential equations can be extended to higher-order systems as well.

Second-order systems

Let

$$\ddot{y} + 2\dot{y} + y = 0,\ y(0^-) = 1,\ \dot{y}(0^-) = 2. \tag{10.108}$$

Taking the Laplace transform of this, and recalling that

$$\mathcal{L}[\dot{y}] = sY(s) - y(0^-), \quad \mathcal{L}[\ddot{y}] = s^2Y(s) - sy(0^-) - \dot{y}(0^-), \tag{10.109}$$

gives that

$$s^2Y(s) - s - 2 + 2sY(s) - 2 + Y(s) = 0 \ \Rightarrow\ Y(s) = \frac{s+4}{s^2+2s+1} = \frac{s}{(s+1)^2} + \frac{4}{(s+1)^2}. \tag{10.110}$$

Now, we know that $\mathcal{L}[t] = 1/s^2$ and also, $\mathcal{L}[x(t)e^{at}] = X(s-a)$, which tells us that

$$\frac{1}{(s+1)^2} \overset{\mathcal{L}^{-1}}{\Longrightarrow} te^{-t}u(t). \tag{10.111}$$

But, what about $s/(s+1)^2$? Recall that $\mathcal{L}[f'] = sX(s) - x(0)$ so if $x(t) = te^{-t}u(t)$ (and, as such $x(0) = 0$), we have that

$$sX(s) = \frac{s}{(s+1)^2} \ \rightarrow_{\mathcal{L}^{-1}}\ \frac{d}{dt}(te^{-t}u(t)) = [e^{-t} - te^{-t}]u(t) = [(1-t)e^{-t}]u(t). \tag{10.112}$$

Summarizing all of this gives that

$$Y(s) = \frac{s}{(s+1)^2} + \frac{4}{(s+1)^2} \ \Rightarrow\ y(t) = [(1-t)e^{-t} + 4te^{-t}]u(t) = (1+3t)e^{-t}u(t). \tag{10.113}$$

The general case

Now, consider the general situation where we are given a Laplace transform $Y(s)$, given by

$$Y(s) = \frac{P(s)}{Q(s)}, \tag{10.114}$$

where $P(s)$ and $Q(s)$ are both polynomials in s. Moreover, assume that the degree of $P(s) = a_ms^m + \ldots + a_0$ is less than the degree of $Q(s) = s^n + b_{n_1}s^{n-1} + \ldots + b_0$, i.e., that $m < n$, as well as the coefficient in front of s^n is 1 in $Q(s)$. We will investigate how to do the PFE for this general situation. And, it turns out that there are two different cases depending on the roots to $Q(s)$.

Case 1: Distinct roots

We start by assuming that all of Q's roots are different. Let the roots be $p_1, \ldots, p_n$, in which case we can write

$$Y(s) = \frac{P(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}. \tag{10.115}$$

If the roots are all indeed different, then there are (possibly complex) constants $A_1, \ldots, A_n$ such that

$$P(s) = \frac{A_1}{s-p_1} + \ldots + \frac{A_n}{s-p_n}. \tag{10.116}$$

As an example, let

$$Y(s) = \frac{1}{s^2 + 2s - 3}. \tag{10.117}$$

We note that the roots to $Q(s)$ are given by

$$s^2 + 2s - 3 = 0 \;\Rightarrow\; s = -1 \pm \sqrt{1+3} = -1 \pm 2 = -3, 1. \tag{10.118}$$

In other words,

$$Y(s) = \frac{1}{(s-1)(s+3)} = \frac{A_1}{s-1} + \frac{A_2}{s+3} = \frac{A_1(s+3) + A_2(s-1)}{(s-1)(s+3)}. \tag{10.119}$$

What we do now is, again, to identify the coefficients:

$$\begin{array}{ll} s^0: & 3A_1 - A_1 = 1 \\ s^1: & A_1 + A_2 = 0 \;\Rightarrow\; A_1 = -A_2. \end{array} \tag{10.120}$$

Plugging $A_1 = -A_2$ into the first equation yields

$$A_1 = \frac{1}{4}, \quad A_2 = \frac{1}{4}, \tag{10.121}$$

i.e.,

$$X(s) = \frac{1}{4(s-1)} - \frac{1}{4(s+3)} \;\Rightarrow\; x(t) = \frac{1}{4}e^t - \frac{1}{4}e^{-3t}. \tag{10.122}$$

What about a slightly more involved example? Let

$$Y(s) = \frac{1}{s^3 + 2s^2 + 5s}. \tag{10.123}$$

Here one root is given by $s = 0$ and the remaining two roots are

$$s^2 + 2s + 5 = 0 \;\Rightarrow\; s = -1 \pm 2j. \tag{10.124}$$

As a consequence, we have

$$\begin{aligned} Y(s) &= \frac{1}{s(s+1-2j)(s+1+2j)} = \frac{A_1}{s} + \frac{A_2}{s+1-2j} + \frac{A_3}{s+1+2j} \\ &= \frac{A_1(s^2+2s+5) + A_2(s^2 + s(1+2j)) + A_3(s^2 + s(1-2j))}{s(s+1-2j)(s+1+2j)}. \end{aligned}$$

Again, identification of the coefficients yeilds:

$$\begin{array}{ll} s^0: & 5A_1 = 1 \;\Rightarrow\; A_1 = \frac{1}{5} \\ s^1: & 2A_1 + (1+2j)A_2 + (1-2j)A_3 = 0 \\ s^2: & A_1 + A_1 + A_3 = 0 \;\Rightarrow\; A_2 = -A_3 - \frac{1}{5}. \end{array} \tag{10.125}$$

Plugging this into the s^1-coefficient equation, we get

$$\frac{2}{5} - (1+2j)A_3 - (1+2j)\frac{1}{5} + (1-2j)A_3 = 0 \;\Rightarrow\; A_3 = -\frac{2+j}{20}, \;\Rightarrow\; A_2 = -\frac{2-j}{20}. \tag{10.126}$$

Collecting up all of these terms gives

$$X(s) = \frac{1}{5}\frac{1}{s} - \frac{2-j}{20}\frac{1}{s-(-1+2j)} - \frac{2+j}{20}\frac{1}{s-(-2-2j)}. \tag{10.127}$$

Taking the inverse Laplace transform yields

$$\begin{aligned} x(t) &= \frac{1}{5}u(t) - \frac{2-j}{20}e^{t(-1+2j)}u(t) - \frac{2+j}{20}e^{t(-1-2j)}u(t) \\ &= \frac{1}{5}u(t) - \frac{1}{20}e^{-t}\left[4\left(\frac{e^{2jt}+e^{-2jt}}{2}\right) + 2\left(\frac{e^{2jt}-e^{-2jt}}{2j}\right)\right]u(t) \\ &= \frac{1}{5}u(t) - \frac{1}{10}e^{-t}\left[2\cos(2t)+\sin(2t)\right]u(t), \quad t \geq 0. \end{aligned} \tag{10.128}$$

Case 2: Repeated roots

As before, let

$$Y(s) = \frac{P(s)}{Q(s)}, \tag{10.129}$$

but now we no longer assume that all of Q's roots are distinct.

As an example, let

$$Y(s) = \frac{1}{(s+1)(s+2)^2}, \quad \Rightarrow \; p_1 = -1, \; p_2 = -2, \; p_3 = -2. \tag{10.130}$$

In this case we have to approach the PFE differently. What we have is really

$$Y(s) = \frac{1}{(s+1)(s+2)^2} = \frac{A_1}{s+1} + \frac{A_{21}+A_{22}s}{(s+2)^2} = \frac{A_1(s^2+4s+4) + A_{21}(s+1) + A_{22}(s^2+s)}{Q(s)}. \tag{10.131}$$

Identifying the coefficients gives

$$\begin{array}{ll} s^0: & 4A_1 + A_{21} = 1 \;\Rightarrow\; A_{21} = -4A_1 + 1 \\ s^1: & 4A_1 + A_{21} + A_22 = 0 \;\Rightarrow\; A_{22} = -1 \\ s^2: & A_1 + A_{22} = 0 \;\Rightarrow\; A_1 = 1 \;\Rightarrow\; A_{21} = -3. \end{array} \tag{10.132}$$

As such,

$$Y(s) = \frac{1}{s+1} - \frac{3+s}{(s+2)^2}. \tag{10.133}$$

Now, recall that

$$\begin{array}{rcl} \dfrac{1}{s^2} & \overset{\mathcal{L}^{-1}}{\Longrightarrow} & tu(t) \\ \dfrac{1}{(s+b)^2} & \overset{\mathcal{L}^{-1}}{\Longrightarrow} & te^{-bt}u(t) \\ \dfrac{s}{(s+b)^2} & \overset{\mathcal{L}^{-1}}{\Longrightarrow} & \dfrac{d}{dt}\left(te^{-bt}u(t)\right) = [e^{-bt} - bte^{-bt}]u(t). \end{array}$$

Summarizing this yields

$$x(t) = [e^{-t} - 3te^{-2t} + 2te^{-2t} - e^{-2t}]u(t) = [e^{-t} - (t+1)e^{-2t}]u(t), \quad t \geq 0. \tag{10.134}$$

Returning to the general case, let $Y(s) = P(s)/Q(s)$, with $\deg(P) < \deg(Q)$. Let $Q(s) = (s-p_1)^{k_1} \cdots (s-p_m)^{k_m}$. Then the PFE looks like

$$Ys) = \frac{A_{1,1} + A_{1,2}s + \ldots + A_{1,k_1-1}s^{k_1-1}}{(s-p_1)^{k_1}} + \ldots + \frac{A_{m,1} + A_{m,2}s + \ldots + A_{m,k_m-1}s^{k_m-1}}{(s-p_m)^{k_m}}. \tag{10.135}$$

The inverse transform of a term like

$$\frac{A_{l,1} + A_{l,2}s + \ldots + A_{l,k_l-1}s^{k_l-1}}{(s-p_l)^{k_l}} \tag{10.136}$$

can be derived as follows:

$$\begin{array}{ccl}
\frac{1}{s} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \mathrm{u}(t) \\
\frac{1}{s^2} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & tu(t) \\
\frac{1}{s^3} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \frac{1}{2}t^2u(t) \\
 & \vdots & \\
\frac{1}{s^k} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \frac{1}{(k-1)!}t^{k-1}u(t) \\
\frac{1}{(s-p)^k} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \frac{1}{(k-1)!}t^{k-1}e^{pt}u(t) \\
\frac{s}{(s-p)^k} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \frac{d}{dt}\left(\frac{1}{(k-1)!}t^{k-1}e^{pt}\right)u(t) \\
\frac{s^2}{(s-p)^k} & \stackrel{\mathcal{L}^{-1}}{\Longrightarrow} & \frac{d^2}{dt^2}\left(\frac{1}{(k-1)!}t^{k-1}e^{pt}\right)u(t) \\
 & \vdots &
\end{array}$$

Although it is reassuring to know that we can solve pretty much any differential equation associated with a LTI system (may require LOTS of work), we actually do not really want to (or need to) go through the PFE unless absolutely necessary since it requires quite a bit of work. Instead, as we will see, what we do want to do is take a few shortcuts that will allow us to not only analyze and characterize the behaviors of input-output systems, but ultimately to control them in effective ways.

10.7 Transfer functions

We now know how to solve linear differential equations with constant coefficients using the Laplace transform and PFE. But the real power behind this formalism derives from these tools' ability to say things about the behaviors of linear time-invariant systems. In particular, in terms of relating the input signals to the output signals.

10.7.1 Input-output systems

Many LTI systems can be written as a linear differential equation with constant coefficients, such as

$$y^n + a_{n-1}y^{n-1} + \ldots + a_1\dot{y} + a_0 = b. \tag{10.137}$$

Since the Laplace transform of the time-derivative of a general signal $f(t)$ is

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-) \tag{10.138}$$

we get, when we take the Laplace transform of the differential equation above, that

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)Y(s) = P_{init}(s) + bX(s), \tag{10.139}$$

where $P_{init}(s)$ is a polynomial of order $n-1$ that involves the initial conditions. We effectively assume that $x(0-) = 0$.

For example, if

$$\ddot{y} + 2\dot{y} - y = 7x, \quad \dot{y}(0^-) = 1, \; y(0^-) = 2, \tag{10.140}$$

we get

$$s^2Y(s) - 2s - 1 + 2sY(s) - 2 - Y(s) = 7X(s) \;\Rightarrow\; (s^2 + 2s - 1)Y(s) = 2s + 3 + 7X(s), \tag{10.141}$$

i.e., $P_{init}(s) = 2s + 3$ in this particular case.

Returning to the general case, we have that

$$Y(s) = \frac{P_{init}(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} + \frac{b}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}X(s). \tag{10.142}$$

If we let

$$Q(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0, \tag{10.143}$$

and let

$$H(s) = \frac{b}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{b}{Q(s)}, \tag{10.144}$$

be the *transfer function* of the system, we can further reduce this to

$$Y(s) = \frac{P_{init}(s)}{Q(s)} + H(s)X(s). \tag{10.145}$$

As a final observation, we note that when the system starts at rest (i.e., all the initial conditions are zero), then we get that

$$Y(s) = H(s)X(s), \tag{10.146}$$

which is known as the *zero-state response*, while $x(t) = 0$ gives

$$Y(s) = \frac{P_{init}(s)}{Q(s)}, \tag{10.147}$$

which is the *zero-input* response. And, it is worth noting that these two responses contribute linearly to the total response, i.e., one can analyze their effect of the system independently of each other. Or, in other words, the total response of a system is the way the system responds to the input if it starts at rest (zero-state) plus the natural drift that is caused by the initial conditions in the absence of control inputs (zero-input). For this reason, we will treat these two different responses independently and, as will be clear further on, we can largely ignore the zero-input response since its contribution to the total response is always just a transient and it will never significantly change anything about how the system behaves.

Modes

In light of the previous discussion, assume that we have a system that is initially at rest (all initial conditions are zero). In that case, we can have

$$Y(s) = H(s)X(s), \tag{10.148}$$

where $H(s)$ tells us how the input affects the output. For this reason, $H(s)$ is known as the *transfer function* to the system since it transfers input signals to output signals.

As an example, let

$$H(s) = \frac{1}{1+s} \tag{10.149}$$

and let $u(t) = \mathrm{u}(t),\ y(0) = 0$, then

$$Y(s) = \frac{1}{s(s+1)}, \tag{10.150}$$

which in turn can be put through the PFE machinery to yield

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} \ \Rightarrow\ y(t) = [1 - e^{-t}]u(t),\ t \geq 0. \tag{10.151}$$

Now, instead, let us pick a different input signal, e.g., let

$$x(t) = e^{-2t} \ \Rightarrow\ X(s) = \frac{1}{s+2} \ \Rightarrow\ Y(s) = \frac{1}{(s+1)(s+2)}. \tag{10.152}$$

Again, PFE tells us that

$$Y(s) = \frac{1}{s+1} - \frac{2}{s+2} \ \Rightarrow\ y(t) = [e^{-t} - e^{-2t}]u(t),\ t \geq 0. \tag{10.153}$$

So, in both of these cases the term $e^{-t}u(t)$ is present in $y(t)$. Where does this term come from? Well, it comes from the system itself, i.e., it is somehow part of what the system does no matter what $X(s)$ is. in fact, for any input $x(t)$, we have

$$Y(s) = \frac{1}{s+1}X(s) \tag{10.154}$$

and the PFE will produce an additive component of $1/(s+1)$ (forget about repeated roots for now), which translates to an additive components of $e^{-t}u(t)$ in $y(t)$ for just about any input $u(t)$. For this reason, we say that $e^{-t}u(t)$ is a *mode* of the system!

There is, of course, nothing special about $e^{-t}u(t)$. For example, if

$$H(s) = \frac{1}{s} \tag{10.155}$$

then for every input

$$Y(s) = \frac{1}{s}X(s) \tag{10.156}$$

and hence $1/s$ will have an additive effect on the output (via PFE). As a consequence, $y(t)$ will have $u(t)$ as a mode.

In general, let the transfer function be given by

$$H(s) = \frac{P(s)}{Q(s)} \tag{10.157}$$

and assume that p is a root of $Q(s)$, i.e., $Q(p) = 0$. For now, assume that p is a distinct root and that $X(s)$ also does not have p as a root. In that case, we can write

$$H(s) = \frac{P(s)}{(s-p)Q'(s)} \;\Rightarrow\; Y(s) = \frac{P(s)X(s)}{Q'(s)} \cdot \frac{1}{s-p}. \tag{10.158}$$

By PFE, we have that

$$Y(s) = \frac{A}{s-p} + \text{stuff} \tag{10.159}$$

i.e., that

$$y(t) = Ae^{pt}u(t) + \mathcal{L}^{-1}[\text{stuff}]. \tag{10.160}$$

As a consequence, $e^{pt}u(t)$ is a mode to the system.

If p is real, we have three distinctly different interpretations of what the system is doing:

1. $p > 0$: In this case, the mode is an increasing exponential and no matter what the other modes are, this "bad" mode will dominate the behavior of the system and $y \to \pm\infty$.
2. $p = 0$: In this case, the mode is 1, i.e., a constant.
3. $p < 0$: In this case, the mode is a decaying exponential and, as $t \to \infty$, the mode will "disappear", i.e., it will be zero and not influence the system.

What about complex poles? Well, if $p = \sigma + j\omega$ then we know that there is another root of $Q(s)$ given by its complex conjugate, i.e., by $\sigma - j\omega$. The output will thus be of the form

$$y(t) = [Ae^{\sigma t}e^{j\omega t} + Be^{\sigma t}e^{-j\omega t}]u(t) + \text{more stuff} = [e^{\sigma t}(C_1\cos(\omega t) + C_2\sin(\omega t))]u(t) + \text{more stuff}, \tag{10.161}$$

for some real constants C_1 and C_2. Since we really only care about asymptotic behavior here, we consider

$$e^{\sigma t}\sin(\omega t)u(t) \tag{10.162}$$

to be the mode of the pair of poles $\sigma \pm j\omega$. The actual output sinuoid would have a an amplitude and phase associated with C_1 and C_2.

Again, three possibilities present themselves here:

1. $\sigma > 0$: In this case, the mode is an oscillation with exponentially increasing amplitude.
2. $\sigma = 0$: Pure oscillation.
3. $\sigma < 0$: Oscillation whose amplitude decays down to zero as $t \to \infty$.

So, just as in the real case, a positive real part corresponds to a "bad" mode, while a negative real part makes the mode vanish for large enough t.

It turns out that this way of thinking about the modes works almost the same when the roots are non-distinct (repeated). In fact, there are two different ways in which roots may be repeated. One is if the root is repeated in $Q(s)$ itself. The second is if the root gets repeated since it is a root of $X(s)$. For example, if $H(s) = 1/s$ and $x(t) = \mathrm{u}(t)$, then

$$Y(s) = \frac{1}{s^2}, \tag{10.163}$$

i.e., $s = 0$ is a repeated root.

Regardless of where the repeated root comes from, as we have seen, the effect of the fact that the root is repeated only translates into the mode being

$$p(t)e^{pt}u(t), \tag{10.164}$$

for some polynomial in t. And, no polynomial is strong enough to overcome an exponential, i.e., if the root has negative real part, then the mode dies down while a positive real part makes it blow up. The only potentially tricky part is when the root has zero real part, as was the case above. In this case, $y(t) = tu(t)$, i.e., it goes off to infinity.

Since the roots to $Q(s)$ are so important, they have their own names, namely *poles*. And, as we will see, these poles will entirely determine the stability properties of the system.

Zero-state and zero-input responses

In light of the mode discussion, we can now revisit the idea of a zero-state and zero-input response. Recall that

$$\text{total response} \;=\; \text{zero-state response} \;+\; \text{zero-input response}, \tag{10.165}$$

or, with math,

$$Y(s) = \frac{P_{init}(s)}{Q(s)} + H(s)X(s). \tag{10.166}$$

But, recall also that

$$H(s) = \frac{P(s)}{Q(s)}, \tag{10.167}$$

so the same modes that are contributed by the zero-state response are usually also contributed by the zero-input response, since these modes are given by the roots to $Q(s)$. We say "usually" since it's feasible that the initial conditions may be set such that a factor of $P_{init}(s)$ cancels one of the factors in $Q(s)$, but it would be rare for such a situation to arise in practice.[4]

As a consequence, if the zero-state response blows up, then so does the zero-input response. Also, if the zero-state response decays down to zero, then so does the zero-input response.

The punchline from this is that, asymptotically, we can simply ignore the initial conditions and focus exclusively on the zero-state response. This means that we will write

$$Y(s) = H(s)X(s), \tag{10.168}$$

which is correct when the system starts at rest. And, even though it is not technically entirely correct when the system does not start at rest, it is still all that we need in order to say things about how the system behaves for large enough t.

The only thing that the initial conditions will in fact contribute with are transients. In other words, the initial conditions may change what the system is doing for a while. But not as $t \to \infty$!

[4]As an example of such a peculiar situation, consider the example of $\ddot{y} + 3\dot{y} + 2y = u(t)$ with the initial conditions $y(0) = 1$, $\dot{y}(0) = 2$ (provided by Matthieu Bloch.)

10.7.2 Stability

So far we have focused on the asymptotic properties of individual modes and on solutions to differential equations. In fact, we have rather informally discussed the notions of asymptotic stability ($y \to 0,\ t \to \infty$), instability ($y \to \pm\infty$) and critical stability (in-between the two). But, we are really not all that interested in systems that only go to zero or infinity. Instead we need to relate this to the input and the key idea is that, as long as the input is bounded, the output should be bounded as well. In fact, this is exactly the stability property we are interested in and we state this as a formal definition:

Definition: An input-output system is BIBO (bounded-in, bounded-out) if a bounded input results in a bounded output.

Now, definitions are nice. But, they do not tell us what we really want to know, namely when a system is BIBO. The key to unlocking this resides with the *poles*. In other words, let $Y(s) = H(s)X(s)$, where the transfer function is

$$H(s) = \frac{P(s)}{Q(s)}, \tag{10.169}$$

and where $p_1, \ldots, p_n$ are the poles, i.e., $Q(p_i) = 0$. Since the PFE analysis means that each such pole contributes a mode to the total response of the system, we know that the modes are given by $e^{p_i t}$ (possibly times a polynomial in t if the poles are repeated), which goes to zero if $\text{Re}(p_i) < 0$ and blows up if $\text{Re}(p_i) > 0$. From this, it is also clear that if a mode blows up, the system is not BIBO. And, if all modes decay down to zero the system is in fact BIBO. It turns out that this is indeed the condition for BIBO, and we state this as a theorem:

Theorem: Let the poles to $H(s)$ be given by $p_1, \ldots, p_n$. The system is BIBO if and only if $Re(p_i) < 0,\ \forall i$.

But wait a minute, what if $\text{Re}(p_i) = 0$? The corresponding modes do not blow up − why is it that such a system is not BIBO. Let's investigate:

First, assume that

$$H(s) = \frac{1}{s}, \tag{10.170}$$

i.e., it has the pole $p = 0$. Now, let $x(t) = \text{u}(t)$, which means that

$$Y(s) = \frac{1}{s^2} \ \Rightarrow \ y(t) = tu(t) \to \infty. \tag{10.171}$$

So, at least in this case, we know that a bounded input (a step) will drive the output off to infinity, i.e., the system is certainly not BIBO. (Note that we really do not care what the initial conditions are since the asymptotic properties of the system do not depend on the initial conditions. So, for the purpose of stability analysis, we only consider the zero-state response.)

But what if the poles lie on the imaginary axis? For example, let

$$H(s) = \frac{1}{s^2+4}. \tag{10.172}$$

Let's try $x(t) = \text{u}(t)$ and see what happens. We get

$$Y(s) = \frac{1}{s^2+4} \cdot \frac{1}{s} = \frac{1}{(s-2j)(s+2j)s} = \frac{A_1}{s} + \frac{A_2}{s-2j} + \frac{A_3}{s+2j}. \tag{10.173}$$

After some calculations, the PFE fields:

$$Y(s)\frac{1}{4}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{1}{1-2j}-\frac{1}{8}\cdot\frac{1}{s+2j}, \tag{10.174}$$

i.e.,

$$y(t)=\left[\frac{1}{4}-\frac{1}{8}e^{2jt}-\frac{1}{8}e^{-2jt}\right]u(t)=\frac{1}{4}[1-\cos(2t)]u(t), \tag{10.175}$$

which does not go off to infinity. So, does this mean that we just disproved the theorem? No! BIBO means that the output has to be bounded for *all* bounded inputs. So, let's try another input. In fact, let

$$u(t)=\sin(2t)u(t). \tag{10.176}$$

This gives

$$Y(s)=\frac{1}{s^2+4}\cdot\frac{2}{s^2+4}=\frac{2}{(s^2+4)^2}=\frac{2}{(s-2j)^2(s+2j)^2}=\frac{A_{11}+A_{12}s}{(s-2j)^2}+\frac{A_{21}+A_{22}s}{(s+2j)^2}. \tag{10.177}$$

Let us not go through the hassle of actually computing the coefficients. Instead, let's go directly for the inverse Laplace transforms:

$$\begin{aligned}\frac{A_{11}}{(s-2j)^2} &\overset{\mathcal{L}^{-1}}{\Longrightarrow} A_{11}te^{2jt}u(t)\\ \frac{A_{21}}{(s-2j)^2} &\overset{\mathcal{L}^{-1}}{\Longrightarrow} A_{21}te^{2jt}u(t)\end{aligned}$$

In fact, let's not even bother with the A_{12} and A_{22} terms, since the contribution from the A_{11} and A_{21} terms combine to a term on the form

$$t\sin(2t+\phi)u(t), \tag{10.178}$$

for some phase angle ϕ. And, this term grows unbounded as $t\to\infty$. Hence the system is not BIBO after all and we can safely trust the theorem.

Note: This rather interesting fact that if the mode is a pure sinusoid $\sin(\omega t)$ then by hitting the system with an input of the same frequency $\sin(\omega t)$, the output grows unbounded. This is known as *resonance*!

10.7.3 Examples

Consider a serial RLC-circuit. The transfer function that connects the voltage over the capacitor V_{out} to the input voltage V_{in} is given by

$$H(s)=\frac{1}{LCs^2+RCs+1}. \tag{10.179}$$

The poles are given by

$$\begin{aligned}&LCs^2+RCs+1=0\\ &s=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}=\frac{-RC\pm\sqrt{R^2C^2-4LC}}{2LC}.\end{aligned}$$

Let us investigate the different parameter values associated with the circuit to see what its stability properties are: If $R^2C^2 - 4LC > 0$ then both poles are negative and real, i.e., the system is BIBO. If $R^2C^2 - 4LC < 0$ then the poles are

$$\frac{-RC \pm j\sqrt{4LC - R^2C^2}}{2LC}, \tag{10.180}$$

i.e., the real part is negative for both poles and, as such, the system is still BIBO as long as $R > 0$. If we remove the resistor, then all of a sudden the poles are purely imaginary and the system is resonant, with resonance frequency

$$\frac{\sqrt{4LC}}{2LC} = \frac{1}{\sqrt{LC}}. \tag{10.181}$$

Now consider a mechanical system given by a mass M connected to an input force u, via a damper d and spring k:

$$M\ddot{y} = -d\dot{y} - ky + u \;\Rightarrow\; H(s) = \frac{1}{Ms^2 + ds + k}. \tag{10.182}$$

The poles are given by

$$s = \frac{-d \pm \sqrt{d^2 - 4Mk}}{2M}. \tag{10.183}$$

Again, we need to untangle two different cases. If $d^2 - 4Mk > 0$ then both poles are negative and real (a so-called over-damped system), while, if $d^2 - 4Mk < 0$ then the poles are

$$s = \frac{-d \pm j\sqrt{4Mk - d^2}}{2M}, \tag{10.184}$$

which has negative real part, i.e., the system oscillates with decaying amplitudes (under-damped system). Again, if $d = 0$ we have the special case of a resonant system, with resonance frequency $\sqrt{k/M}$.

10.7.4 Asymptotic behavior

One of the key aspects of being able to establish BIBO is that we no longer have to add the disclaimer "as long as the limit exists" when applying final value theorem. As long as the system is BIBO and the input is bounded then we can indeed apply the FVT without having to feel nervous.

Chapter 11

Frequency Responses of Second-order Systems

There are two common ways of notating the denominators of system functions of second-order LTI systems:

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + \frac{\omega_n}{Q}s + \omega_n^2. \tag{11.1}$$

In both notations, ω_n represents the system's *undamped natural frequency.* The first formula in (11.1) specifies a *damping ratio* ζ. The second specifies a *quality factor* Q. They are trivially related by $Q = 1/(2\zeta)$ and $\zeta = 1/(2Q)$. The damping ratio convention tends to be preferred by control systems engineers; they are often worried about ζ being too low (equivalently, Q being too high). Filter designers tend to use the quality factor[1] convention; when designing a bandpass filter, they usually desire high Q (equivalently, low ζ). Because ζ offers a bit of simplification in terms of typography, we primarily use it in our derivations, but we will present the main results in terms of both conventions.

Using the quadratic formula, the poles of the system, i.e. the roots of $D(n)$, are found to lie at

$$s_p = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}\right). \tag{11.2}$$

For $\zeta = 1$ (or $Q = 1/2$), the square root term disappears, and both poles lie at ω_n. For $\zeta > 1$ (or $Q < 1/2$), the term under the radical is positive, and both poles lie on the negative real axis. For $\zeta < 1$ (or $Q > 1/2$), the term in the under the radical is negative, and the poles appear as complex conjugate pair.

For $\zeta \geq 1$, i.e. $Q \leq 1/2$, a second-order filter may be implemented as a cascade of two first-order filters with half-power cutoff frequencies given by the $\omega_{co} = -s_p$, where the poles s_p are given by (11.2). However, this is not the case for $\zeta < 1$, i.e. $Q > 1/2$, since it does not make sense to talk about a first-order filter with a complex cutoff; a specific second-order filter structure is needed.

[1] Designers of electronic musical instruments with second-order filters, such as the Oberheim SEM, the Yamaha CS-80, often use terms like *resonance*, *regeneration*, or *emphasis* for Q. These names are often also applied to other feedback-related aspects of other musical filters, such as a fourth-order filter structure initially popularized by Robert Moog; for such complicated structures, these names may not map exactly to Q as we have defined it here.

The next few sections will focus on frequency responses. To prepare, we substitute $s = j\omega$ in (11.1) to yield

$$D(j\omega) = (j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2 = -\omega^2 + j2\zeta\omega_n\omega + \omega_n^2 = (\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega = (\omega_n^2 - \omega^2) + j\frac{\omega_n}{Q}\omega. \quad (11.3)$$

The parenthesis in (11.3) are not really necessarily, but they suggest a convenient logical grouping. Taking the squared magnitude of this yields

$$\begin{aligned} |D(j\omega)|^2 = D(j\omega)D^*(j\omega) &= [(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega][(\omega_n^2 - \omega^2) - j2\zeta\omega_n\omega] & (11.4) \\ &= (\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2 = (\omega_n^2 - \omega^2)^2 + \frac{\omega_n^2}{Q^2}\omega^2 & (11.5) \\ &= \omega^4 + (4\zeta^2 - 2)\omega_n^2\omega^2 + \omega_n^4 = \omega^4 + \left(\frac{1}{Q^2} - 2\right)\omega_n^2\omega^2 + \omega_n^4. & (11.6) \end{aligned}$$

The compact forms in (11.5) and the more spread out forms in (11.6) will both come in handy.

Without loss of generality, we can write the following canonical lowpass, bandpass, and highpass transfer functions:

$$H_{2LP}(s) = \frac{\omega_n^2}{D(s)}, \quad H_{2BP}(s) = \frac{2\zeta\omega_n s}{D(s)} = \frac{(\omega_n/Q)s}{D(s)}, \quad H_{2HP}(s) = \frac{s^2}{D(s)}. \quad (11.7)$$

The constants in the numerators of (11.7) have been chosen so that the filters have unity gain at "DC" for the lowpass filter, the peak frequency (defined later) for the bandpass filter, and "infinite frequency" for the highpass filter.

Because the analyses of the lowpass and highpass filters are similar, we will present them first before moving on to the bandpass filter.

11.1 Second-order lowpass filter

A second-order lowpass transfer function may be written as

$$H_{2LP}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}. \quad (11.8)$$

Just looking at (11.8), it is clear that $H_{LP}(j0) = 1$ and $H_{LP}(j\omega) \to 0$ as $\omega \to 0$.

The squared magnitude of the frequency response is

$$|H_{2LP}(j\omega)|^2 = \frac{\omega_n^4}{|D(j\omega)|^2}. \quad (11.9)$$

The magnitude response will exhibit a peak for certain values of ζ (or Q). To locate the peak, it is sufficient to find a minimizer of $1/|H_{2LP}(j\omega)|^2$. Since the numerator is a constant, we may set the derivative of the denominator of (11.9) with respect to ω^2 (symbolically treating ω^2 as a variable) equal to zero, using the form in (11.6), yielding

$$\begin{aligned} \frac{d}{d(\omega^2)}\{(\omega^2)^2 + (4\zeta^2 - 2)\omega_n^2\omega^2 + \omega_n^4\} &= 0 & (11.10) \\ 2\omega^2 + (4\zeta^2 - 2)\omega_n^2 &= 0 & (11.11) \\ \omega^2 &= (1 - 2\zeta^2)\omega_n^2. & (11.12) \end{aligned}$$

So, if a peak exists, it appears at

$$\omega_{LP,r} = \omega_n \sqrt{1 - 2\zeta^2} = \omega_n \sqrt{1 - \frac{1}{2Q^2}}. \tag{11.13}$$

Note that (11.13) does not make sense if the quantity under the radical is negative. If $\zeta \geq 1/\sqrt{2}$, i.e. $Q \leq 1/\sqrt{2}$, then the magnitude of the frequency response monotonically decreases with increasing ω, as illustrated in the top row, left column of Figure **??** for $\zeta = 1$ (the critically damped case). Otherwise, it will manifest a single peak at $\omega_{LP,r}$, as illustrated in top row, right column of Figure **??** for $\zeta = 1/4$. Notice that $\omega_{LP,r} \to \omega_0$ *from the left* as $\zeta \to 0$, i.e. $Q \to \infty$. As a consistency check, notice that plugging $\zeta = Q = 1/\sqrt{2}$ into (11.13) yields $\omega_{LP,r} = 0$. This case is shown in the top row, middle column of Figure **??**.

The magnitude of the peak can readily be calculated by substituting $\omega_{LP,r}$ into $|H_{2LP}(j\omega)|^2$.

$$\begin{aligned}
|H_{2LP}(j\omega_{LP,r})|^2 &= \frac{\omega_n^4}{(\omega_n^2 - \omega_{LP,r}^2)^2 + (2\zeta\omega_n\omega_{LP,r})^2} && (11.14)\\
&= \frac{\omega_n^4}{\omega_n^4(1 - (1 - 2\zeta^2))^2 + 4\zeta^2\omega_n^4(1 - 2\zeta^2)} && (11.15)\\
&= \frac{1}{4\zeta^4 + 4\zeta^2 - 8\zeta^4} && (11.16)\\
&= \frac{1}{4\zeta^2(1 - \zeta^2)}. && (11.17)
\end{aligned}$$

The resulting magnitude is only a function of ζ (or Q):

$$|H_{2LP}(j\omega_{LP,r})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}. \tag{11.18}$$

11.2 Second-order highpass filter

A second-order lowpass transfer function may be written as

$$H_{2HP}(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}. \tag{11.19}$$

Just looking at (11.19), it is clear that $H_{HP}(j0) = 0$ and $H_{HP}(j\omega) \to 1$ as $\omega \to \infty$.

The squared magnitude of the frequency response is

$$|H_{2LP}(j\omega)|^2 = \frac{\omega^4}{|D(j\omega)|^2}. \tag{11.20}$$

The magnitude response will exhibit a peak for certain values of ζ (or Q). As in the lowpass case, we can find the peak of (11.20), if it exists, by finding the minimizer of

$$\frac{1}{|H_{2LP}(j\omega)|^2} = \frac{\omega^4 + (4\zeta^2 - 2)\omega_n^2\omega^2 + \omega_n^4}{\omega^4} = 1 + (4\zeta^2 - 2)\omega_n^2\omega^{-2} + \omega_n^4\omega^{-4}. \tag{11.21}$$

We used the form in (11.6). If we think of ω^{-2} as a variable, we can set the derivative of (11.21) with respect to ω^{-2} equal to zero, yielding:

$$\frac{d}{d(\omega^{-2})}\{1+(4\zeta^2-2)\omega_n^2\omega^{-2}+\omega_n^4\omega^{-4}\} = 0 \quad (11.22)$$

$$(4\zeta^2-2)\omega_n^2+2\omega_n^4\omega^{-2} = 0 \quad (11.23)$$

$$2\omega_n^2\omega^{-2} = 2-4\zeta^2 \quad (11.24)$$

$$\omega^{-2} = \frac{1-2\zeta^2}{\omega_n^2} \quad (11.25)$$

$$\omega^2 = \frac{\omega_n^2}{1-2\zeta^2}. \quad (11.26)$$

So, if a peak exists, it appears at

$$\omega_{HP,r} = \omega_n\frac{1}{\sqrt{1-2\zeta^2}} = \omega_n\frac{1}{\sqrt{1-\frac{1}{2Q^2}}}. \quad (11.27)$$

Note that (11.27) does not make sense if the quantity under the radical is negative. If $\zeta \geq 1/\sqrt{2}$, i.e. $Q \leq 1/\sqrt{2}$, then the magnitude of the frequency response monotonically increases with increasing ω, as illustrated in the middle row, left column of Figure **??** for $\zeta = 1$ (the critically damped case). Otherwise, it will manifest a single peak at $\omega_{HP,r}$, as illustrated in middle row, right column of for $\zeta = 1/4$. Notice that $\omega_{HP,r} \to \omega_n$ *from the right* as $\zeta \to 0$, i.e. $Q \to \infty$. Intuitively speaking, if you plug $\zeta = Q = 1/\sqrt{2}$, $\omega_{HP,r}$ is at "infinity." This case is shown in the middle row, middle column of Figure **??**. Comparing (11.27) with its lowpass equivalent (11.13), we see that the multipliers of ω_n possess an elegant reciprocal symmetry.

The magnitude of the peak can be calculated by substituting $\omega_{HP,r}$ into $|H_{2HP}(j\omega)|^2$ as was done for the lowpass filter to yield,

$$|H_{2HP}(j\omega_{HP,r})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}, \quad (11.28)$$

which is exactly the same as for the lowpass filter.

11.3 Second-order bandpass filter

A second-order bandpass transfer function may be written as

$$H_{2BP}(s) = \frac{2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{(\omega_n/Q)s}{s^2+\frac{\omega_n}{Q}s+\omega_n^2}. \quad (11.29)$$

Looking at (11.29), it is clear that $H_{BP}(j0) = 0$ and $H_{BP}(j\omega) \to 0$ as $\omega \to \infty$.

The squared magnitude of the frequency response is

$$|H_{2BP}(j\omega)|^2 = \frac{4\zeta^2\omega_n^2\omega^2}{|D(j\omega)|^2} = \frac{(\omega_n^2/Q^2)\omega^2}{|D(j\omega)|^2}. \quad (11.30)$$

Using the compact denominator form of (11.5) and dividing the numerator and denominator by $\omega_2\omega_n^2$ leads to some insight:

$$|H_{2BP}(j\omega)|^2 = \frac{4\zeta^2\omega_n^2\omega^2}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} = \frac{4\zeta^2}{\left(\frac{\omega_n}{\omega} - \frac{\omega}{\omega_n}\right)^2 + 4\zeta^2}. \tag{11.31}$$

This form lets us find the peak value of the frequency response *without needing to take a derivative.* Looking at (11.31), we see that there is a constant in the numerator, so to make (11.31) as large as possible, it suffices to the denominator of (11.31) as small as possible. Both terms in the denominator are nonnegative, and the second term is a constant, so making the first term equal to zero best we can hope for, which is easily achieved by setting $\omega = \omega_n$, yielding unity gain at that *peak frequency.*

Bandwidth: Unlike the highpass and lowpass responses we discussed earlier, the peak of the bandpass response is fixed at ω_n. The form used above is normalized so to have unity gain at the peak frequency. So any magnitude below that peak will appear at two frequencies, one at a frequency above ω_n and one at a frequency below ω_n. Let's denote these as ω_U and ω_L, respectively. No matter what horizontal cut you make through the magnitude response graph, it turns out that

$$\omega_n = \sqrt{\omega_L\omega_U}. \tag{11.32}$$

This can be seen through the following steps:

$$\begin{aligned}
\left(\frac{\omega_n}{\omega_L} - \frac{\omega_L}{\omega_n}\right) &= -\left(\frac{\omega_n}{\omega_U} - \frac{\omega_U}{\omega_n}\right). && (11.33)\\
\frac{\omega_n}{\omega_L} + \frac{\omega_n}{\omega_U} &= \frac{\omega_U}{\omega_n} + \frac{\omega_L}{\omega_n}. && (11.34)\\
\omega_n^2\left[\frac{1}{\omega_L} + \frac{1}{\omega_U}\right] &= \omega_U + \omega_L. && (11.35)\\
\omega_n^2\left[\frac{\omega_L + \omega_U}{\omega_L\omega_U}\right] &= \omega_U + \omega_L. && (11.36)\\
\omega_n^2 &= \omega_U\omega_L. && (11.37)
\end{aligned}$$

Inspired by Bode plots, (11.32) has a natural interpretation if we take the logarithm of both sides:

$$\ln\omega_n = \frac{\ln\omega_L + \ln\omega_U}{2}. \tag{11.38}$$

The peak frequency is evenly spaced between matching upper and lower frequency points *when viewed on a log frequency scale.* So ω_n is the "center frequency" in this logarithmic sense; it is the geometric mean of ω_L and ω_U. This nicely matches how humans perceive pitch, since we perceive audio frequencies logarithmically; for instance, 110, 220, 440, and 880 Hz are all considered "A" notes.

We now consider a specific horizontal cut, the half-power cut. Let $\omega_{L,1/2}$ and $\omega_{U,1/2}$ denote the frequencies where $|H(j\omega) = 1/\sqrt{2} = 0.707$. If we define the bandwidth as $BW = \omega_{U,1/2} - \omega_{L,1/2}$, then it turns out that $Q = \omega_n/BW$. So for a fixed center frequency ω_n, higher Q gives lower bandwidth and lower Q gives higher bandwidth. But notice that Q is normalized by ω_n. This makes Q a particularly useful control for musicians. Because of the logarithmic way that humans perceive pitch, a fixed bandwidth centered around high frequency will "sound" much tighter than the same bandwidth centered around a low frequency. By providing musicians direct control of Q instead of BW, the actual bandwidth automatically adapts as the musician changes ω_n, so the effective bandwidth *perceived* by the listener seems to remain the same as ω_n is "swept."

Variations: If a filter design specification calls for $Q \leq 1/2$, the designer can just build a one-pole lowpass filter and follow it with a one-pole highpass filter or vice-versa (sometimes there's good practical reasons for picking a particular order). However, most bandpass filter specifications of interest call for a high Q value, which would require a specific second-order circuit structure.

The form of the bandpass response used through this section was normalized to have unity gain at the center frequency. Another common way to express it has a gain of Q at the center frequency, so increasing Q increases the gain as well as narrowing the peak, as shown in the lower right panel of Figure 11.2. Many circuit realizations found in textbooks use that convention. Some circuit implementations, such as the state variable filter, provide both an output with gain Q and a unity gain output.

11.4 Second-order Butterworth filters

The $\zeta = Q = 1/\sqrt{2}$ cases, in which the poles are at 45 degree angles in the s-place, represent special cases of a class of filters called "Butterworth," in this case a second-order Butterworth. Butterworth filters have the steepest possible cutoffs without having undesired peaks near the cutoff in the frequency response. These undesired peaks are called "ripple." In general, sharper cutoffs can often be achieved at the expense of introducing ripple in the frequency response.

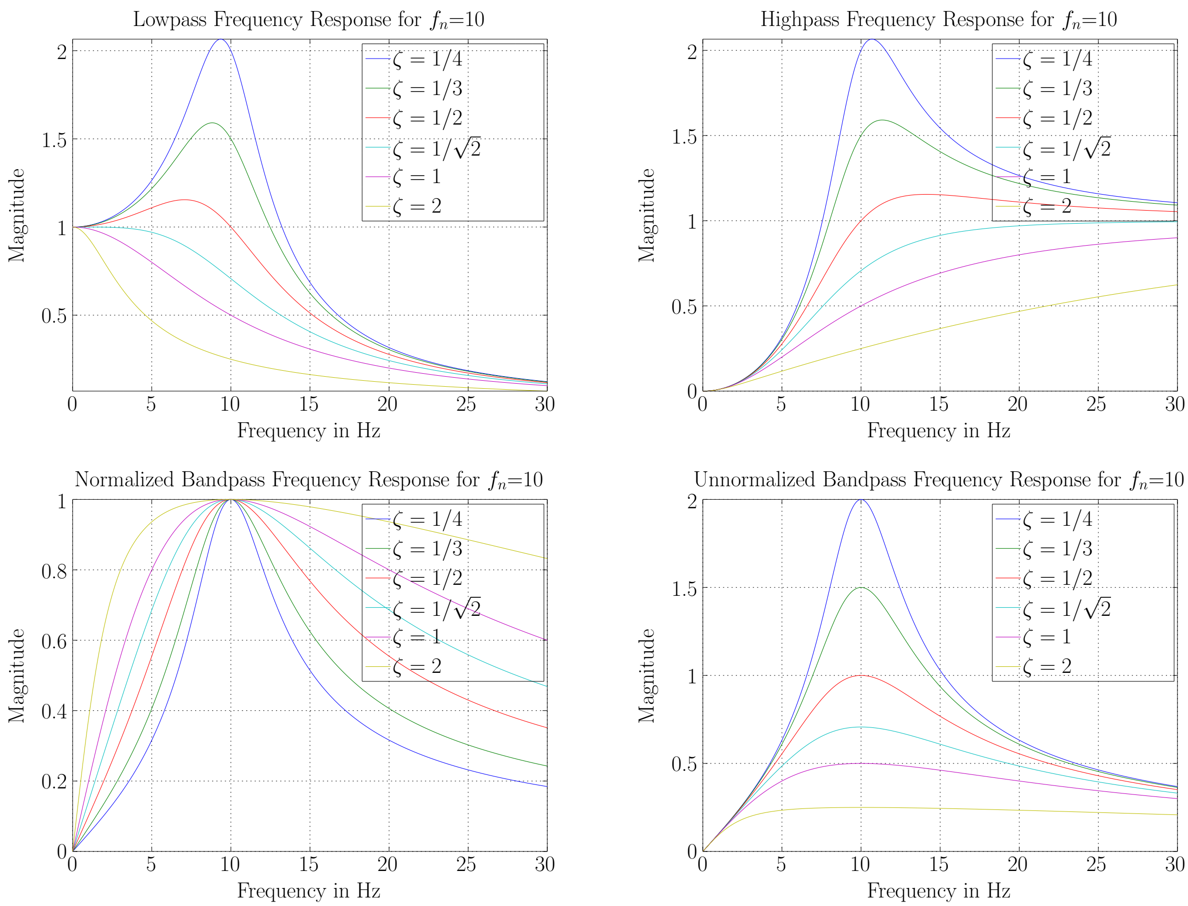


Figure 11.1: Magnitudes of example second-order frequency responses with natural frequency of $f_n = 10$ Hz (i.e., $\omega_n = 20\pi$ radians/second): lowpass (upper left), highpass (upper right), normalized bandpass (lower left), and unnormalized bandpass (upper right). The five lines in each graph correspond to damping factors $zeta = 2$ (underdamped), $\zeta = 1$ (critically damped), $1\sqrt{2}$ (Butterworth), 1/2, 1/3, and 1/4.

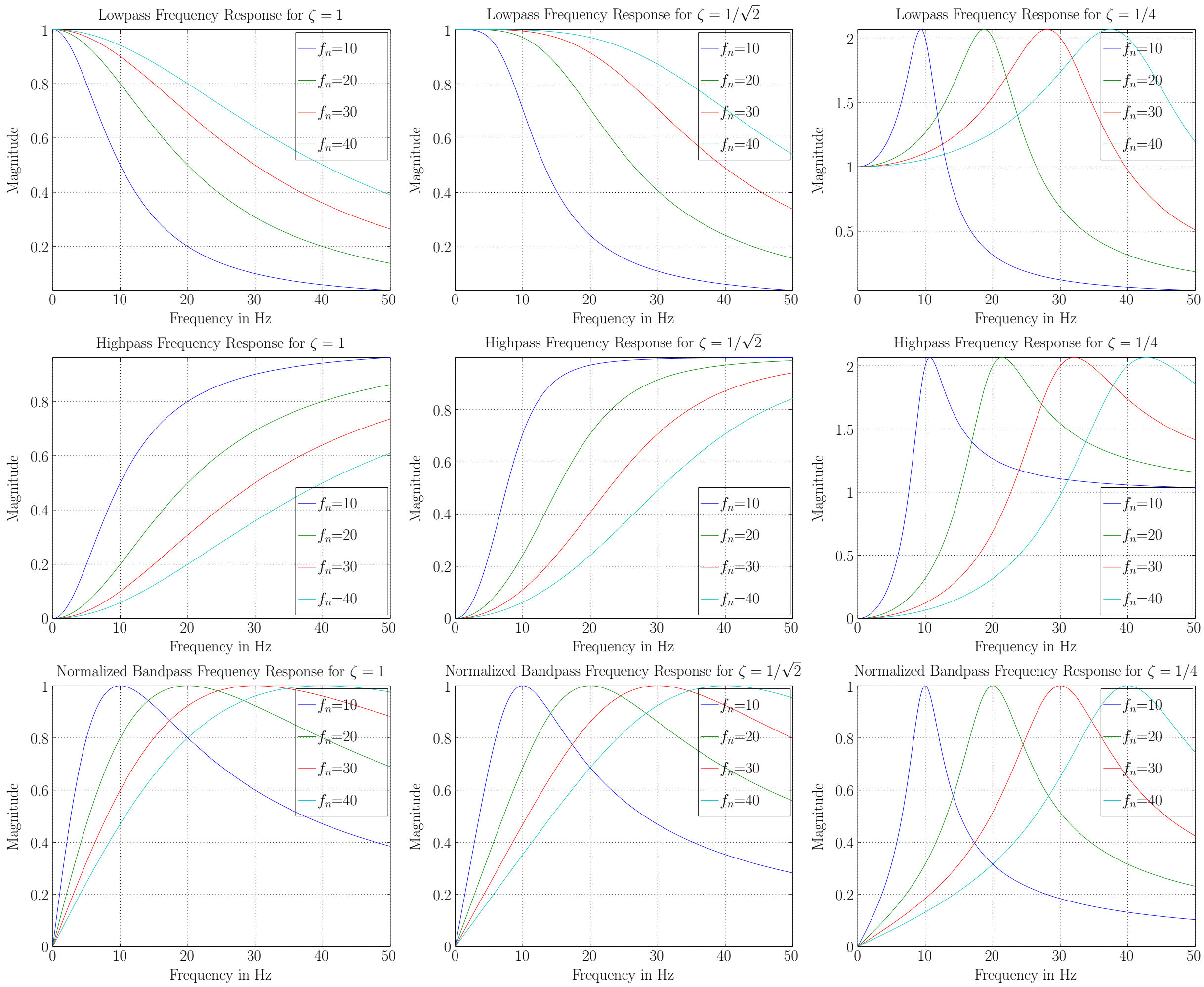


Figure 11.2: Magnitudes of example second-order frequency responses. The top, middle, and bottom rows correspond to lowpass, highpass, and bandpass functions, respectively. From left to right, the columns correspond to damping factors $\zeta = 1$ (critically damped), $1\sqrt{2}$ (Butterworth), and 1/4. The four lines in each graph correspond to natural frequencies (in Hz) of $f_n = 10$, 20, 30, and 40 Hz; the corresponding natural frequencies in radians/second are given by $\omega_n = 2\pi f_n$.

Chapter 12

Connecting the s and z Planes

12.1 Rise of the z-transforms

Recall our model of the sampling process explored in Section (9.1.3), in which we multiplied a continuous-time signal with an impulse train:

$$x_s(t) = \sum_{n=0}^{\infty} x(nT_s)\delta(t - nT_s). \tag{12.1}$$

Here, we will repeat this analysis from the viewpoint of Laplace transforms. Since this text deals exclusively with unilateral Laplace transforms, we let the lower limit of the summation be $n = 0$ instead of $n = -\infty$ as in the earlier chapter using Fourier transforms; we also restrict $x(t)$ to be zero for $t < 0$.

Using our mild abuse of notation $x[n] = x(nT_s)$, the Laplace transform of (12.1) is

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x[n]\delta(t - nT_s)e^{-st}dt = \sum_{n=0}^{\infty} x[n] \int_0^{\infty} \delta(t - nT_s)e^{-st} \tag{12.2}$$

$$= \sum_{n=0}^{\infty} x[n]e^{-s(T_s n)} = \sum_{n=0}^{\infty} x[n](e^{-sT_s})^n. \tag{12.3}$$

Notice that for any integer k,

$$X_s(s + jk\omega_s) = X_s\left(s + jk\frac{2\pi}{T_s}\right) = \sum_{n=0}^{\infty} x[n]\exp\left(-\left[s + j\frac{2\pi}{T_s}\right]T_s n\right) = \sum_{n=0}^{\infty} x[n]e^{-s(T_s n)}e^{-j2\pi n}. \tag{12.4}$$

Since n is an integer, the $\exp(-j2\pi n)$ factor in the last expression in (12.4) is 1. Hence, that expression looks like the first expression in (12.3). Thus,

$$X_s(s + jk\omega_s) = X_s(s). \tag{12.5}$$

Just as sampling in the time domain with a sampling rate T_s induced spectral replicas with spacing ω_s in the Fourier domain, such sampling induces replicas of horizontal strips of the s-plane with a vertical spacing of ω_s. This "aliasing" makes the Laplace transform a bit annoying to deal with in the case of sampled signals.

It is convenient to make the substitution $z = \exp(sT_s)$ in the last expression of (12.3), and reconsider the Laplace transform as a function of this new variable z:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \quad (12.6)$$

This is our old friend, the unilateral *z-transform* of $x[n]$, which you will recall from your earlier studies of discrete-time systems.

The mapping $z = \exp(sT_s)$ uniquely maps the horizontal strip of the s-plane consisting of $-\omega_s/2 < \Im m\{s\} < \omega_s/2$ to the entire z-plane. The expression $s = \ln(z)/T_s$ allows us to more-or-less map every point of the z-plane back to the horizontal strip of the s-plane.

Seeing the logarithm in the equation $s = \ln(z)/T_s$ may seem a bit strange, since it involves taking logarithms of negative numbers, which your high school algebra teacher probably told you could not be done, and taking logarithms of complex numbers in general, which your high school algebra teacher may never have even thought about. However, most computational mathematics packages will happily return a value of the natural logarithm for complex numbers, and it will come to pass that taking the natural exponent of that natural logarithm will give you your original complex number back. Two quirks that lead to the "more-or-less" qualifier in the previous paragraph:

- Since $\exp(a + jb) = \exp(a + j[b + 2\pi k])$ for real a and b and integer k, the inverse of the exponential is not unique. By convention, $\ln(z)$ is presumed to return the "principal value" lying between $-\pi$ and π, so for our purposes, $s = \ln(z)/T_s$ maps points in the z-plane to the horizontal strip in the s-plane surrounding the real axis.

- Technically speaking, there is no value a for which $\exp(a) = 0$. Most computational mathematics packages will return some representation of "$-\infty$," instead of producing some kind of error message, when asked to compute $\ln(0)$. For the purposes of linear system analysis, that is not a bad way to think about it.

The above discussion hammers home the point that there is nothing particularly magical about the z-transform; it can be viewed as a special case of the Laplace transform with some convenient customized notation.

12.2 Converting a continuous-time filter to a discrete-time approximation

Suppose we had an "analog" filter that we wanted to emulate using "digital signal processing." We put those terms in quotes since the real issue is the *continuous-time* nature of our "prototype" filter and the *discrete-time* nature of its implementation; one could imagine implementing the discrete-time filter using analog multipliers and clocked bucket-brigade devices instead of using a microprocessor with analog-to-digital and digital-to-analog converters (although few engineers in their right mind would do so).

Suppose we have a continuous-time filter specified by a set of poles in the s-plane, generically labeled s_p. We could create an approximation of this by cascading a continuous-to-discrete converter, a discrete-time LTI system, and a discrete-to-continuous converter, with the converters running at a sample rate T_s. There are many approaches to crafting the discrete-time LTI system in the middle. An intuitive approach based on our "derivation" of the z-transform would be to use the expression $z_p = \exp(s_pT_s)$ to map poles in the

s-plane to poles in the z-plane. This is sometimes referred to as the "matched z-transform method" or the "pole-zero mapping" method. If the original system function $H(s)$ has no zeros, this procedure also corresponds to the "impulse invariance method." These methods and others are thoroughly covered in texts on digital signal processing. Here, we merely hope to give you a taste of this kinds of procedure.

Let us refer to the system function of the discrete-time LTI system with the mapped poles as $H_{\text{pzm}}(z)$. Recall that the frequency response of a discrete-time system can be found by evaluating its system function at $z = \exp(j\hat{\omega})$, where the normalized frequency is $\hat{\omega} = \omega/f_s$, just as the frequency response of a continuous-time system can be found by evaluating its system function at $s = j\omega$. The *effective* frequency response of the full cascade, which includes C-D and D-C converters surrounding the system $H_{\text{pzm}}(z)$, is

$$H_{\text{eff}}(j\omega) = \begin{cases} H_{\text{pzm}}(e^{j\omega/f_s}) & \text{for } |w| < \omega_s/2 \\ \text{undefined} & \text{otherwise} \end{cases} . \tag{12.7}$$

Input frequencies of half the sample rate or above will result in aliasing, which will cause the cascade to no longer act linearly, which in turn defenestrates the idea of a frequency response. Even for frequencies below half the sample rate, the effective frequency response of the discrete-time implementation, $H_{\text{eff}}(j\omega)$, is never going to exactly match that of the original system, $H(s)$. If we are lucky, we might feel justified in writing $H_{\text{eff}}(j\omega) \approx H(j\omega)$ for $|w| < \omega_s/2$.

Another common technique for implementing approximations of continuous-time filters using discrete-time filters is the *bilinear transformation.* We will leave a discussion of this technique to more specialized references.

12.2.1 Example: converting a Butterworth filter

A lowpass Butterworth filter[1] of order N (i.e. having N poles) with a "cutoff frequency" of ω_c has the magnitude-squared frequency response

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}. \tag{12.8}$$

Butterworth filters of a given order exhibit the steepest possible cutoff that may be achieved without any "ripples" in the magnitude of the frequency response (i.e., the magnitude of the frequency response is a monotonic function). Higher orders exhibit steeper cutoffs.

Second-order example

For $N = 2$, we have the classic second-order lowpass filter we studied previously, with $\omega_n = \omega_c$ and $\zeta = 1/\sqrt{2}$. The poles lie at

$$s_p = -\frac{\omega_c}{\sqrt{2}} \pm j\frac{\omega_c}{\sqrt{2}}. \tag{12.9}$$

[1] In addition to Butterworth filters, there are Type-I and Type-II Chebyshev filters and elliptic filters. These terms refer to forms of transfer functions, not particular physical implementations. There are entire books on analog filter design, and "Signals and Systems" texts often include entire chapters going through various filters and their realization with operational amplifiers, resistors, capacitors, and inductors. Here, we are just using the Butterworth filter as an example of the process of approximating an analog filter with a digital filter.

One could implement such a filter using operations, capacitors, and resistors. Alternatively, we could create an approximation using digital signal processing, for instance, using the pole-zero mapping method:

$$z_p = \exp(s_p T_s) = \exp\left(s_p \frac{2\pi}{\omega_s}\right) = \exp\left(\omega_c \left[-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right] \frac{2\pi}{\omega_s}\right). \quad (12.10)$$

Suppose we set the sample rate of our DSP-based implementation to be four times the desired cutoff frequency, leading to $\omega_c = \omega_s/4$ and

$$z_p = \exp\left(\frac{\omega_s}{4}\left[-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right]\frac{2\pi}{\omega_s}\right) = \exp\left(\frac{\pi}{2\sqrt{2}}[-1 \pm j]\right) \quad (12.11)$$

$$= \exp\left(-\frac{\pi}{2\sqrt{2}}\right)\exp\left(\pm j\frac{\pi}{2\sqrt{2}}\right) = 0.3293e^{\pm j1.1107}. \quad (12.12)$$

The z-plane poles lie at $\pm 63.64°$.

The discrete-time filter has the transfer function

$$H_{pzm}(z) = \frac{1}{(1 - z_{p_1}z^{-1})(1 - z_{p_2}z^{-1})} \quad (12.13)$$

$$= \frac{1}{(1 - 0.3293e^{j1.1107}z^{-1})(1 - 0.3293e^{-j1.1107}z^{-1})} \quad (12.14)$$

$$= \frac{1}{1 - 0.2924z^{-1} + 0.1084z^{-2}}, \quad (12.15)$$

corresponding to the difference equation

$$y[n] = 0.2924y[n-1] - 0.1084y[n-2] + x[n]. \quad (12.16)$$

12.2.2 Third-order example

A third-order, lowpass Butterworth filter with cutoff ω_c has three s-plane poles, consisting of one conjugate pair at $\omega_c \exp(\pm j2\pi/3)$ and a single real pole at $-\omega_c$. Suppose $\omega_c = 2\pi f_c$, where $f_c = 6$ kHz.

The transfer function of the continuous-time filter is

$$H(s) = \frac{\omega_c^3}{(s - p_1)(s - p_2)(s - p_3)}.$$

The upper left panel of Figure 12.1 shows the magnitude the frequency response of this continuous time filter, $|H(j\omega)| = |H(j2\pi f)|$, for $0 \leq f \leq 30$ kHz. Note that $|H(j\omega_c)|$ is approximately $1/\sqrt{2} = 0.707$.

Using the "pole-zero mapping" method, namely $z_p = \exp(s_p T_s)$, where T_s is the sampling period, the z-plane poles of the discrete-time filter that approximates this Butterworth filter for a sample rate of $f_s = 40$ kHz are at $0.42759 \pm 0.45478j$ and 0.38966.

Denote the discrete-time pole-zero mapped system as

$$H_{pzm}(z) = \frac{C}{(1 - z_{p_1}z^{-1})(1 - z_{p_2}z^{-1})(1 - z_{p_3}z^{-1})} \quad (12.17)$$

$$= \frac{Cz^3}{(z - z_{p_1})(z - z_{p_2})(z - z_{p_3})}. \quad (12.18)$$

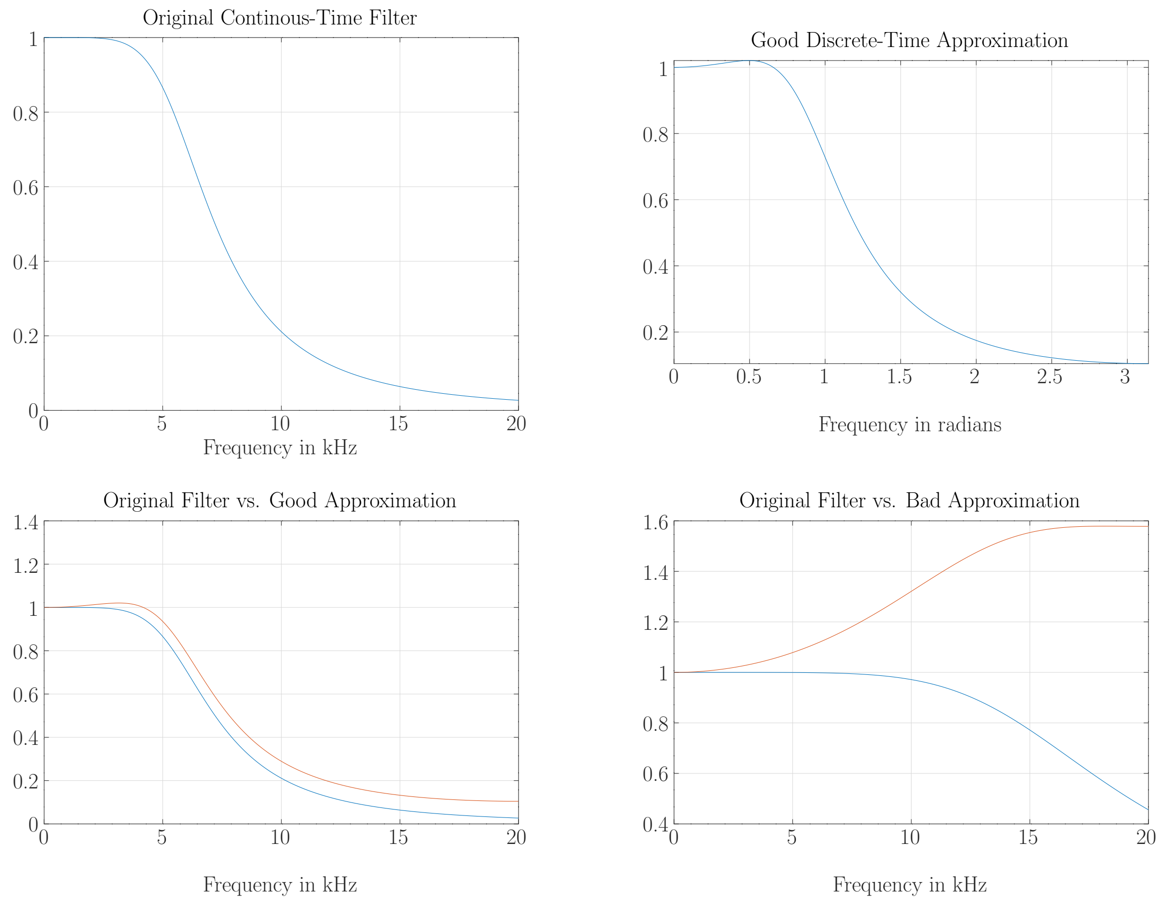


Figure 12.1: Captain goes here.

The first version is the most convenient for multiplying out and turning into a difference equation to actually implement the filter. The second form is the most conceptualizing frequency responses.

We introduced the C is a factor to force the frequency response of the continuous-time system being approximated and the discrete-time implementation to match up at some desired frequency. In this example, we the choose $C = 0.32622$, which makes $H_{pzm}(e^{j0}) = H_{pzm}(1)$ equal 1; i.e., we'll set up our discrete-time approximation to have unity gain at "D.C.," just as the original continuous-time system does.

The upper right panel of Figure 12.1 shows $|H_{pzm}(e^{j\hat{\omega}})|$ for $0\hat{\omega} \leq \pi$.

The lower left panel of Figure 12.1 shows the frequency response of the original continuous-time filter, $|H(j2\pi f)|$ for f ranging from 0 to 20 kHz against $|H_{\text{eff}}(j2\pi f)| = |H_{pzm}(e^{j2\pi f/f_s})|$, which is the magnitude of the frequency response of the discrete-time approximation consisting of a cascade of a continuous-to-discrete converter, the IIR (infinite impulse response) filter implied by (12.17).s (Remember that because of aliasing, the expression for $|H_{\text{eff}}(j2\pi f)|$ is not meaningful for f larger than half the sample rate of $f_s = 40$ kHz.) The two systems match reasonably well.

If we repeat the above exercise with $f_c = 16$ kHz, we find the new z-plane poles $-0.16205 \pm 0.23397j$ and 0.081. Here, f_c is close to half of the sampling frequency, so the match between the original continuous-time filter and its continuous-time approximation is catastrophically poor, as shown in the lower right corner of Figure 12.1

Chapter 13

Step Responses of Second-order Systems

13.1 Second-order lowpass filter

A second-order lowpass transfer function may be written as

$$H_{2LP(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}. \tag{13.1}$$

The poles are located at $\omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$.

13.1.1 Overdamped lowpass response

If $\zeta > 1$, the poles p_1, p_2 are distinct and on the negative real axis. The partial fraction expansion of the step response has the form

$$Y_{2LP}(s) = \frac{H_{2LP}(s)}{s} = \frac{c_0}{s} + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2}. \tag{13.2}$$

Using the residue method, the coefficient associated with the term containing the step is

$$c_0 = Y_{2LP}(s)s|_{s=0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}|_{s=0} = \frac{\omega^2}{\omega^2} = 1. \tag{13.3}$$

We could also quickly find c_0 by using the idea of frequency response and computing $H(j\omega)$ at $\omega = 0$. The resulting step response is

$$y_{2LP}(t) = u(t) + c_1 e^{p_1 t}u(t) + c_2 e^{p_2 t}u(t). \tag{13.4}$$

Let $p_1 = \omega_n(-\zeta + \sqrt{\zeta^2 - 1})$ and $p_2 = \omega_n(-\zeta - \sqrt{\zeta^2 - 1})$ The term with the pole closest to the imaginary axis (in this case p_1, which one might call the "slowest" term, tends to dominate the behavior of the response.

The residue method yields

$$c_1 \quad = \quad Y_{2LP}(s)s|_{s=p_1} = \left.\frac{\omega_n^2}{s(s-p_2)}\right|_{s=p_1} = \frac{\omega_n^2}{p_1(p_1-p_2)} = \frac{1/2}{\zeta^2 - 1 - \zeta\sqrt{\zeta^2-1}}. \tag{13.5}$$

$$c_2 \quad = \quad Y_{2LP}(s)s|_{s=p_1} = \left.\frac{\omega_n^2}{s(s-p_1)}\right|_{s=p_2} = \frac{\omega_n^2}{p_2(p_2-p_1)} = \frac{1/2}{\zeta^2 - 1 + \zeta\sqrt{\zeta^2-1}}. \tag{13.6}$$

Examples are shown via two of the lines in the left panel of Figure 13.1.

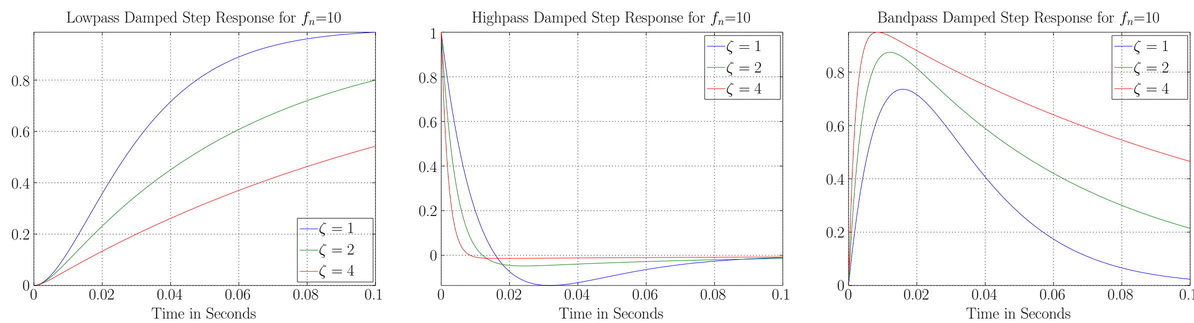


Figure 13.1: Step responses for damped lowpass (left panel), highpass (center panel), and bandpass (right panel) second-order systems with $f_n = 10$ Hz; $\omega_n = 2\pi f_n$. The three lines in each graph correspond to damping factors of $\zeta = 1$ (critically damped), $\zeta = 2$, and $\zeta = 4$.

13.1.2 Critically damped lowpass response

If $\zeta = 1$, a double pole is present at $-\omega_n$:

$$H_{2LP(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}. \tag{13.7}$$

$$Y_{2LP}(s) = \frac{H_{2LP}(s)}{s} = \frac{c_0}{s} + \frac{c_1}{s+\omega_n} + \frac{c_2}{(s+\omega_n)^2}. \tag{13.8}$$

As before, it is easy to determine that $c_1 = 1$. Using the residue method to determine c_1 and c_2, we have

$$c_2 = Y_{2LP}(s)(s+\omega_n)^2|_{s=-\omega_n} = \frac{\omega_n^2}{s}|_{s=-\omega_n} = -\omega_n. \tag{13.9}$$

$$c_1 = \frac{d}{dt}[Y_{2LP}(s)(s+\omega_n)^2]|_{s=-\omega_n} = -\frac{\omega_n^2}{s^2}|_{s=-\omega_n} = -1. \tag{13.10}$$

The resulting step response is

$$y_{2LP}(t) = u(t) - \exp(-\omega_n t)u(t) - \omega_n t\exp(-\omega_n t)u(t) = [1 - (1+\omega_n t)e^{-\omega_n t}]u(t). \tag{13.11}$$

This has a slower response compared with a first order system with its pole at the same location.

Examples are shown in left panel of Figure 13.2 and via one of the lines in the left panel of Figure 13.1.

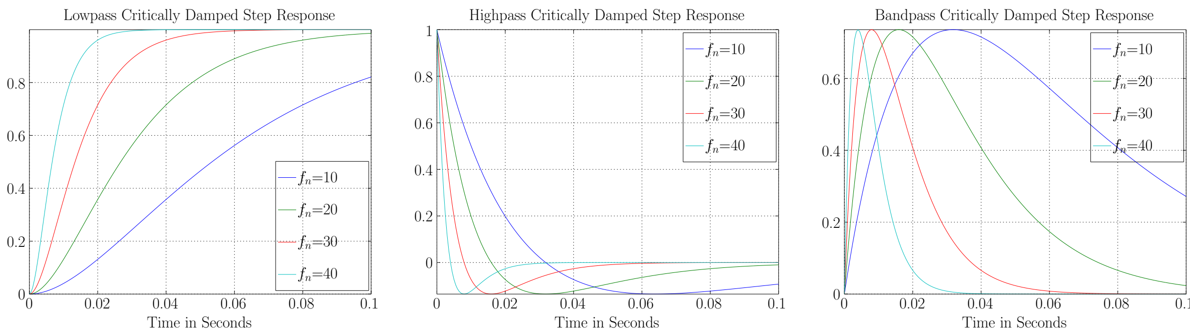


Figure 13.2: Step responses for critically damped lowpass (left panel), highpass (center panel), and bandpass (right panel) second-order systems. The four lines in each graph correspond to natural frequencies of $f_n =$ 5, 10, 20, and 40 Hz; the corresponding natural frequencies in radians/second are given by $\omega_n = 2\pi f_n$.

13.1.3 Underdamped lowpass response

If $\zeta < 1$, then the poles constitute a complex conjugate pair. Defining the *damped frequency* $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the poles are at $-\zeta\omega_n \pm j\omega_d$. We can rewrite the transfer function as

$$H_{2LP}(s) = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_d^2}. \tag{13.12}$$

The output may be written as the partial fraction expansion

$$Y_{2LP}(s) = H_{2LP}(s) = \frac{\omega_n^2}{s[(s+\zeta\omega_n)^2 + \omega_d^2]} = \frac{c_0}{s} + \frac{c_1 s + c_2}{(s+\zeta\omega_n)^2 + \omega_d^2}. \tag{13.13}$$

As before, a quick analysis of the frequency response at DC reveals that $c_0 = 1$. Plugging this into (13.13) gives us

$$Y_{2LP}(s) = \frac{1}{s} + \frac{c_1 s + c_2}{(s+\zeta\omega_n)^2 + \omega_d^2}. \tag{13.14}$$

Multiplying both sides of (13.16) by the denominator of the left hand side yields

$$\omega_n^2 = [(s+\zeta\omega_n)^2 + \omega_d^2] + c_1 s^2 + c_2. \tag{13.15}$$

Equating coefficients of terms with s^2 yields $0 = 1 + c_1$, hence $c_1 = -1$. Equating coefficients of terms with s yields $0 = 2\zeta\omega_n + c_2$, hence $c_2 = -2\zeta\omega_n$. Plugging the coefficients into (13.13) yields

$$Y_{2LP}(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}. \tag{13.16}$$

Cleverly rewriting of the numerator of the second term of (13.16) as $(s + \zeta\omega_n) + (\zeta\omega_n/\omega_d)\omega_d$ puts $Y_{2LP}(s)$ in a form amenable to the trigonometric functions in our Laplace transform tables:

$$
\begin{aligned}
y_{2LP}(t) &= u(t) - e^{-\zeta\omega_n t}\cos(\omega_d t)u(t) - \zeta\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)u(t) && (13.17)\\
&= \left\{1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\left[\frac{\omega_d}{\omega_n}\cos(\omega_d t)u(t) + \zeta\sin(\omega_d t)\right]\right\}u(t) && (13.18)\\
&= \left\{1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\cos(\omega_d t)u(t) + \zeta\sin(\omega_d t)\right]\right\}u(t). && (13.19)
\end{aligned}
$$

To write (13.19), we used the definition $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Leaving the step response in a form with a cosine and a sine of the same frequency might give the reader the mistaken impression that the output has two sinusoids. We can rewrite it as a single sinusoidal term using "phasor addition," representing the cosine term using the phasor $\sqrt{1-\zeta^2}$ and the sine term using the phasor $-j\zeta$. We want to these complex numbers together and convert the result polar form. Notice that $(\sqrt{1-\zeta^2})^2 + \zeta^2 = 1$, so the resulting phasor has unit length, and the angle is $\phi = \arctan(-\zeta/\sqrt{1-\zeta^2})$.

Hence, we can write (13.19) as

$$
y_{2LP}(t) = \left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)\right]u(t), \text{ where } \phi = \arctan\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\right). \qquad (13.20)
$$

We have generally recommended representing generic real sinusoids using the cosine function, as in (13.20). Most textbook authors seem to prefer a representation in terms of the sine function. We can write

$$
\begin{aligned}
y_{2LP}(t) &= \left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi + \pi/2)\right]u(t) && (13.21)\\
&= \left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)\right]u(t), \text{ where } \varphi = \arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right). && (13.22)
\end{aligned}
$$

To create (13.22), we defined $\varphi = \phi + \pi/2$ and used the obscure trigonometric identity $\arctan(\alpha) + \pi/2 = -\arctan(1/\alpha) = \arctan(-1/\alpha)$. Examples are shown in the left column of Figure 13.3.

13.2 Second-order highpass filter

A second-order highpass transfer function may be written as

$$
H_{2HP(s)} = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}. \qquad (13.23)
$$

The Laplace transform of its step response is

$$
Y_{2HP(s)} = \frac{H_{2HP(s)}}{s} = \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \qquad (13.24)
$$

According to the final value theorem, the step response converges to

$$
\lim_{t\to\infty} y_{2HP}(t) = \lim_{s\to 0} sY_{2HP}(s) = \lim_{s\to 0} s\frac{H_{2HP}(s)}{s} = 0. \qquad (13.25)
$$

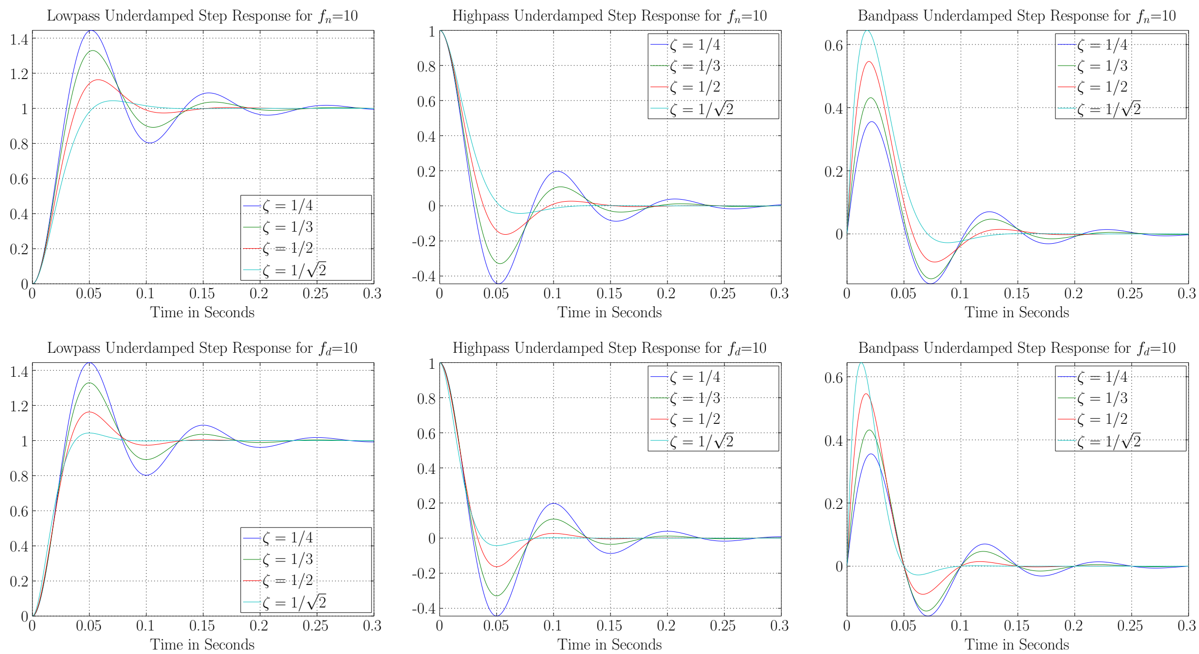


Figure 13.3: Step responses for underdamped lowpass (left column), highpass (center column), and bandpass (right column) second-order systems. The four lines in each graph correspond to damping factors of $\zeta = 1/4$, $\zeta = 1/3$, $\zeta = 1/2$, and $\zeta = 1/\sqrt{2}$ (Butterworth). In the top row, the natural frequency is $f_n = 10$ Hz. In the bottom row, the damped frequency of every line is $f_d = 10$ Hz; to achieve this, we set the natural frequency to $f_n = f_d/\sqrt{1-\zeta^2}$, so each line uses a different f_n.

It's also clear that (13.25) must hold because this highpass filter completely blocks DC.

The initial value theorem tells us that

$$y_{2HP}(0) = \lim_{s\to\infty} sY_{2HP}(s) = \lim_{s\to\infty} s\frac{H_{2HP}(s)}{s} = 1. \tag{13.26}$$

13.2.1 Overdamped highpass response

If $\zeta > 1$, the system has two real poles, and the PFE of the step response has the form

$$Y_{2HP}(s) = \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2}, \tag{13.27}$$

which has the inverse Laplace transform

$$y_{2HP}(t) = c_1 e^{p_1 t}u(t) + c_2 e^{p_2 t}u(t). \tag{13.28}$$

Since (13.45) is a sum of two decaying exponentials, clearly $\lim_{t\to\infty} y(t) = 0$, confirming (13.25).

Let $p_1 = \omega_n(-\zeta + \sqrt{\zeta^2-1})$ and $p_2 = \omega_n(-\zeta - \sqrt{\zeta^2-1})$ The term with the pole closest to the imaginary axis (in this case p_1, which one might call the "slowest" term, tends to dominate the behavior of the response.

The residue method yields

$$c_1 = Y_{2LP}(s)s|_{s=p_1} = \left.\frac{s}{s-p_2}\right|_{s=p_1} = \frac{p_1}{p_1-p_2} = \frac{1}{2}\left[-\frac{\zeta}{\sqrt{\zeta^2-1}}+1\right]. \tag{13.29}$$

$$c_2 = Y_{2LP}(s)s|_{s=p_1} = \left.\frac{s}{s-p_1}\right|_{s=p_2} = \frac{p_2}{p_2-p_1} = \frac{1}{2}\left[\frac{\zeta}{\sqrt{\zeta^2-1}}+1\right]. \tag{13.30}$$

Examples are shown via two of the lines in the middle panel of Figure 13.1.

13.2.2 Critically damped highpass response

If $\zeta = 1$, the system has repeated poles, and the PFE of the step response has the form

$$Y_{2HP}(s) = \frac{s}{(s+\omega_n)^2} = \frac{c_1}{s+\omega_n} + \frac{c_2}{(s+\omega_n)^2}. \tag{13.31}$$

The residue method readily obtains $c_2 = s|_{s=-\omega_n} = -\omega_n$ and $c_1 = 1$, so

$$y(t) = e^{-\omega_n t}u(t) - \omega_n t e^{-\omega_n t)}u(t) \tag{13.32}$$

Clearly $\lim_{t\to\infty} y(t) = 0$, which corroborates (13.25). The step response starts at $y(0) = 1$ and ends at 0, but it does not do so monotonically. It does not have "ripples" in the oscillatory sense, but it does undershoot before returning to the steady state value of 0. Setting $y(t)$ equal to zero for $t > 0$ yields $e^{-\omega_n t} - \omega_n t e^{-\omega_n t)} = 0$. Since $e^{-\omega_n t} > 0$, there is no danger of dividing $e^{-\omega_n t}$ it out of the equation, yielding $1 - \omega_n t = 0$. Hence, we see there is a zero crossing at $t = 1/\omega_n$.

Examples are shown in center panel of Figure 13.2 and via one of the lines in the center panel of Figure 13.1.

13.2.3 Underdamped highpass response

If $\zeta < 1$, the system has complex poles. We can write

$$Y_{2HP}(s) = \frac{(s+\zeta\omega_n) - \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} \tag{13.33}$$

to make it easier to see the application of the trigonometric functions in our Laplace transform tables:

$$y_{2HP}(t) = e^{-\zeta\omega_n t}\cos(\omega_d t)u(t) - \zeta\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)u(t) \tag{13.34}$$

$$= e^{-\zeta\omega_n t}\left[\cos(\omega_d t)u(t) + \zeta\frac{\omega_n}{\omega_d}\sin(\omega_d t)\right]u(t) \tag{13.35}$$

$$= \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)u(t), \tag{13.36}$$

where $\varphi = \arctan(\sqrt{1-\zeta^2}/\zeta)$ as before. Examples are shown in the center column of Figure 13.3.

13.3 Second-order bandpass filter

A second-order bandpass transfer function may be written as

$$H_{2BP(s)} = \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{(\omega_n^2/Q)s}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}. \tag{13.37}$$

According to the final value theorem, the step response converges to

$$\lim_{t\to\infty} y_{2BP}(t) = \lim_{s\to 0} sY_{2BP}(s) = \lim_{s\to 0} s\frac{H_{2BP}(s)}{s} = 0. \tag{13.38}$$

It's also clear that (13.37) must hold because this bandpass filter completely blocks DC.

The initial value theorem tells us that

$$y_{2BP}(0) = \lim_{s\to\infty} sY_{2BP}(s) = \lim_{s\to\infty} s\frac{H_{2BP}(s)}{s} = 0. \tag{13.39}$$

13.3.1 Overdamped bandpass response

If $\zeta > 1$, the system has two real poles, and the PFE of the step response has the form

$$Y_{2BP}(s) = \frac{2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2}, \tag{13.40}$$

which has the inverse Laplace transform

$$y_{2BP}(t) = c_1 e^{p_1 t}u(t) + c_2 e^{p_2 t}u(t). \tag{13.41}$$

Since (13.41) is a sum of two decaying exponentials, clearly $\lim_{t\to\infty} y(t) = 0$, confirming (13.38). Without specifically computing c_1 and c_2, note that since (13.39) tells that $y_{2BP}(0) = 0$, it must be the case that $c_1 = -c_2$. The curve of $y_{2BP}(t)$ rises and then falls, with the "fast" pole (the one furthest from the origin, associated with a positive PFE coefficient) controlling the rise time and the "slow" pole (the one closer to the origin, associated with a negative PFE coefficient) controlling the fall time. The residue method yields

$$c_1 = Y_{2LP}(s)s|_{s=p_1} = \left.\frac{2\zeta\omega_n}{s - p_2}\right|_{s=p_1} = \frac{2\zeta\omega_n}{p_1 - p_2} = \frac{\zeta}{\sqrt{\zeta^2 - 1}}. \tag{13.42}$$

$$c_2 = Y_{2LP}(s)s|_{s=p_1} = \left.\frac{s}{s - p_1}\right|_{s=p_2} = \frac{2\zeta\omega_n}{p_2 - p_1} = -\frac{\zeta}{\sqrt{\zeta^2 - 1}}. \tag{13.43}$$

Examples are shown via two of the lines in the right panel of Figure 13.1.

13.3.2 Critically damped bandpass response

If $\zeta = 1$, there is a double real pole, and

$$Y_{2BP}(s) = \frac{2\omega_n}{(s + \omega_n)^2}, \tag{13.44}$$

which has the inverse Laplace transform

$$y_{2BP}(t) = 2\omega_n t e^{-\omega_n t} u(t). \tag{13.45}$$

As with the overdamped case, this curve rises from and then falls back to zero, with a peak at $t = 1/\omega_n$, which can be found by taking setting the derivative of (13.45) for $t > 0$ equal to zero, yielding $-\omega_n t e^{-\omega_n t} + e^{-\omega_n t} = 0$. Since $e^{-\omega_n t} > 0$, it is safe to divide it through, yielding $-\omega_n t + 1 = 0$. The peak value is $y_{2BP}(1/\omega_n) = 2e^{-1} \approx 0.736$).

Examples are shown in right panel of Figure 13.2 and via one of the lines in the right panel of Figure 13.1.

13.3.3 Underdamped bandpass response

If $\zeta < 1$, there poles are complex, and the Laplace transform of the step response is

$$Y_{2BP}(s) = \frac{2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}, \tag{13.46}$$

which has the inverse transform of

$$\begin{aligned} y_{2BP}(t) &= \frac{2\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) u(t) && (13.47) \\ &= \frac{2\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) u(t) && (13.48) \\ &= \frac{2\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) u(t). && (13.49) \end{aligned}$$

Examples are shown in the right column of Figure 13.3.

13.4 A few observations

Meditate upon the following notes about these second-order responses:

- The step response has "ripples," i.e. oscillates, if it is underdamped at all $\zeta < 1$, but the lowpass and highpass forms only have a resonant peak in the frequency domain if $\zeta < 1/\sqrt{2}$.
- The resonant frequency $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$ of the second order lowpass filter is always less than $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the frequency of oscillation in the time domain. For the high pass filter, $\omega_r = \frac{\omega_n}{\sqrt{1-2\zeta^2}}$ is always greater than both ω_n and ω_d.

Chapter 14

Circuit Analysis via Laplace Transforms

14.1 Laplace-domain circuit models

14.1.1 Resistors

$$v_R(t) = Ri_R(t) \tag{14.1}$$

$$V_R(s) = RI_R(s) = Z_R I(s), \tag{14.2}$$

where $Z_R = R$, the impedance of a resistor, is admittedly not the most interesting impedance in the world.

14.1.2 Capacitors

The fundamental voltage-current relationship of a capacitor is usually first presented as:

$$i_C(t) = C\frac{d}{dt}v_C(t) \tag{14.3}$$

Taking the Laplace transform of this equation yields

$$I_C(s) = C[sV_C(s) - v_C(0^-)] = CsV_C(s) - Cv_C(0^-) = \frac{V_C(s)}{Z_C(s)} - Cv_C(0^-), \tag{14.4}$$

where $Z_C(s) = 1/(Cs)$ is the impedance of a capacitor. This equation can be rearranged to yield an expression for $V_C(s)$:

$$V_C(s) = \frac{I_C(s)}{Cs} + \frac{v_C(0^-)}{s} = Z_C(s)I_C(s) + \frac{v_C(0^-)}{s}. \tag{14.5}$$

Another approach to deriving (14.5) is to divide the time domain expression (14.3) by C and then integrate both sides:

$$\frac{1}{C}\int_{0^-}^{t} i_C(\tau)d\tau = \int_{0^-}^{t} \frac{d}{d\tau}v_C(\tau)d\tau = v_C(t) - v_C(0^-), \text{ for } t \geq 0. \tag{14.6}$$

$$v_C(t) = \frac{1}{C}\int_{0^-}^{t} i_C(\tau)d\tau + v_C(0^-), \text{ for } t \geq 0. \tag{14.7}$$

Taking the unilateral Laplace transform of (14.7) yields (14.5).

Kirchoff's voltage law (KVL) says that voltages in *series* add, while Kirchoff's current law (KCL) says that currents in *parallel* add. Using KCL and (14.4), we can formulate a parallel-structure Laplace-domain equivalent for capacitors consisting of a capacitive impedance and an impulsive current source with a weight of $Cv_C(0^-)$. Using KVL and (14.5), we can formulate a series-structure equivalent consisting of a capacitive impedance and a constant $v_C(0^-)$ voltage source. Because of the minus sign in front of the $Cv_C(0^-)$ term in 14.9, the current source in the parallel model has its arrow running opposite the direction of the current arrow in the passive current convention of the capacitor.

14.1.3 Inductors

The fundamental voltage-current relationship of an inductor is usually first presented as:

$$v_L(t) = L\frac{d}{dt}i_L(t). \tag{14.8}$$

Taking the Laplace transform of this equation yields

$$V_L(s) = L[sI_L(s) - i_L(0^-)] = LsI_L(s) - Li_L(0^-) = Z_L(s)I_L(s) - Li_L(0^-), \tag{14.9}$$

where $Z_L(s) = Ls$ is the impedance of a capacitor. This equation can be rearranged to yield an expression for $I_L(s)$:

$$I_L(s) = \frac{V_L(s)}{Ls} + \frac{i_L(0^-)}{s} = \frac{V_L(s)}{Z_L(s)} + \frac{i_L(0^-)}{s}. \tag{14.10}$$

Another approach to deriving (14.10) is to divide the time domain expression (14.8) by L and then integrate both sides:

$$\frac{1}{L}\int_{0^-}^{t} v_L(\tau)d\tau = \int_{-0}^{t} \frac{d}{d\tau}i_L(\tau)d\tau = i_L(t) - i_L(0^-), \text{ for } t \geq 0. \tag{14.11}$$

$$i_L(t) = \frac{1}{L}\int_{0^-}^{t} v_L(\tau)d\tau + i_L(0^-), \text{ for } t \geq 0. \tag{14.12}$$

Taking the unilateral Laplace transform of (14.12) yields (14.10).

Using KVL and (14.9), we can formulate a series-structure Laplace-domain equivalent for inductors consisting of an inductive impedance and an impulsive voltage source with a weight of $Li_L(0^-)$. Using KCL and (14.10), we can formulate a parallel-structure equivalent consisting of an inductive impedance and a constant $i_L(0^-)$ current source. Because of the minus sign in front of the $Li_L(0^-)$ term in (14.9), the voltage source in the series model has its + and − signs set opposite of the + and − signs in the passive voltage convention of the inductor.

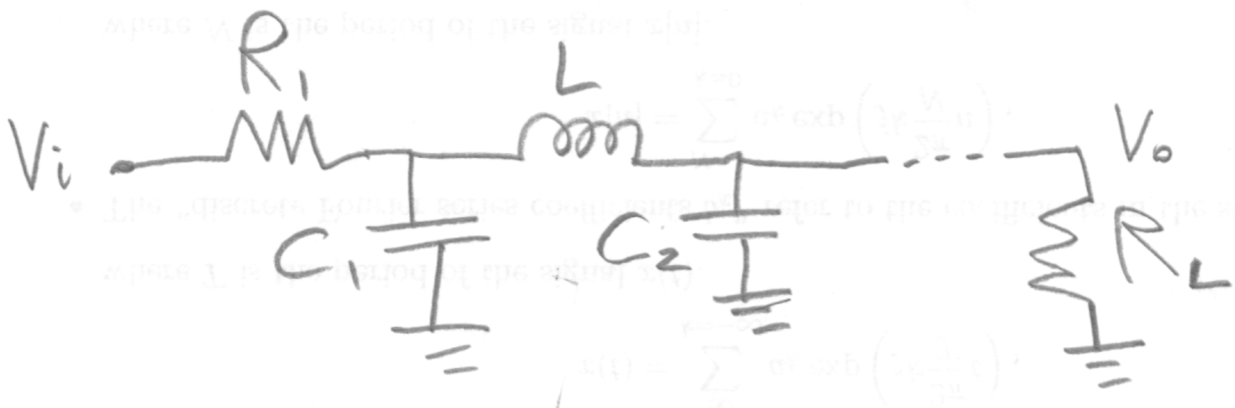


Figure 14.1: CLC pi filter.

14.2

14.3 Pi filters

Pi filters are so named because of their topological resemblance to the Greek letter π, but not because of any particularly deep connection to the number 3.14159 *ad infinitum.* They are often used in high-voltage DC power supplies.

14.3.1 CLC pi filters

In Figure 14.1, $v_i(t)$ is assumed to be an ideal voltage source measured with respect to ground. In a power supply application, $v_i(t)$ is typically a rectified sinusoid, although this filter structure has many other potential uses. The capacitors and the inductor form the pi filter. The resistor represents the nonideal output resistance of the voltage source. In a power supply R_1 typically represents the effective resistance of the rectifier diodes, and it might also include some resistance intentionally added to act as a cheap secondary fuse and to prevent too much current from flooding into C_1 too quickly on when the supply is switched on. R_L represents the load resistance; for the remainder of this section, we will assume it is infinite.

There are several approaches you could take to a problem like this. Here, we first replace the subcircuit consisting of the input voltage source, R_1, and C_1 with its Thévenin equivalent. Voltage division readily yields the Thévenin voltage

$$V_{th}(s) = V_i(s)\frac{1/(C_1 s)}{R_1 + 1/(C_1 s)} = V_i(s)\frac{1}{R_1 C_1 s + 1}. \tag{14.13}$$

To find the Thévenin impedance, we set $V_i = 0$ (shorting the independent voltage source) and take the parallel combination of R_1 and C_1:

$$R_{th}(s) = \frac{R_1/(C_1 s)}{R_1 + 1/(C_1 s)} = \frac{R_1}{R_1 C_1 s + 1}. \tag{14.14}$$

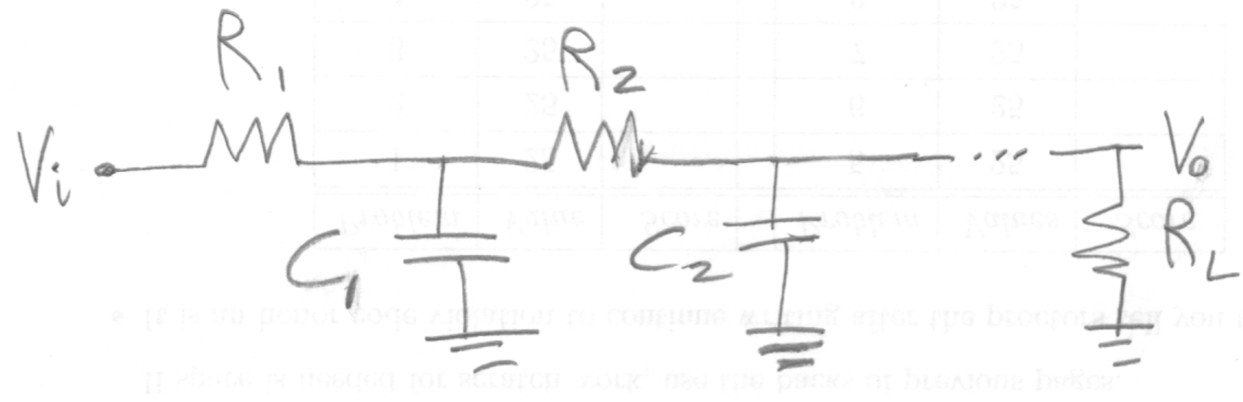


Figure 14.2: CRC pi filter.

Combining the Thévenin impedance (14.14) with the inductance in series yields

$$R_{th}(s) + Ls = \frac{R_1}{R_1C_1s+1} + Ls. \tag{14.15}$$

The impedance computed in (14.15) forms a voltage divider with C_2:

$$\frac{1/(C_2s)}{\frac{R_1}{R_1C_1s+1} + Ls + \frac{1}{C_2s}} = \frac{1/(R_1C_1LC_2)}{\frac{s}{LC_1}\cdot\frac{1}{R_1C_1s+1} + \frac{s^2}{R_1C_1} + \frac{1}{R_1C_1LC_2}}. \tag{14.16}$$

Applying the voltage division factor (14.22) to the Thévenin voltage (14.13) yields

$$\begin{aligned} V_o(s) &= V_i(s)\frac{1}{R_1C_1s+1}\cdot\frac{1/(R_1C_1LC_2)}{\frac{s}{LC_1}\cdot\frac{1}{R_1C_1s+1} + \frac{s^2}{R_1C_1} + \frac{1}{R_1C_1LC_2}} && (14.17)\\ &= V_i(s)\frac{1/(R_1C_1LC_2)}{\frac{s}{LC_1} + (R_1C_1s+1)\left(\frac{s^2}{R_1C_1} + \frac{1}{R_1C_1LC_2}\right)} && (14.18)\\ &= V_i(s)\frac{1/(R_1C_1LC_2)}{\frac{s}{LC_1} + s^3 + \frac{s^2}{R_1C_1} + \frac{s}{LC_2} + \frac{1}{LR_1C_1C_2}} && (14.19)\\ &= V_i(s)\frac{1/(R_1C_1LC_2)}{s^3 + \frac{s^2}{R_1C_1} + s\left(\frac{1}{LC_1} + \frac{1}{LC_2}\right) + \frac{1}{R_1C_1LC_2}}. && (14.20)\end{aligned}$$

The transfer function of this filter is the expression in (14.20) to the right of $V_i(s)$. It could also be derived via the node-voltage method; no particular method is necessarily faster of easier than the others.

14.3.2 CRC pi filters

The inductors needed in Figure 14.1 can be sometimes be large, heavy, and expensive. An alternative is to replace the inductor with a resistor, as shown in Figure 14.2. This results in cheaper, smaller, and lighter hardware, with the downside of less filtering of undesired frequencies and less DC voltage sat the output.

The derivation of the resulting transfer function can follow along the lines of the derivation in Section 14.3.1. We retain the Thévenin voltage (14.13) and Thévenin impedance (14.14) computed for the CLC filter.

Combining the Thévenin impedance (14.14) in series with the new resistor R_2 (which replaces the inductor) yields

$$R_{th}(s) + R_2 = \frac{R_1}{R_1 C_1 s + 1} + R_2. \tag{14.21}$$

The impedance computed in (14.21) forms a voltage divider with C_2:

$$\frac{1/(C_2 s)}{\frac{R_1}{R_1 C_1 s+1} + R_2 + \frac{1}{C_2 s}} = \frac{1/(R_1 C_1 R_2 C_2)}{\frac{s}{R_2 C_1} \cdot \frac{1}{R_1 C_1 s+1} + \frac{s}{R_1 C_1} + \frac{1}{R_1 C_1 R_2 C_2}}. \tag{14.22}$$

Applying the voltage division factor (14.22) to the Thévenin voltage (14.13) yields

$$\begin{aligned}
V_o(s) &= V_i(s) \frac{1}{R_1 C_1 s + 1} \cdot \frac{1/(R_1 C_1 R_2 C_2)}{\frac{s}{R_2 C_1} \cdot \frac{1}{R_1 C_1 s+1} + \frac{s}{R_1 C_1} + \frac{1}{R_1 C_1 R_2 C_2}} && (14.23)\\
&= V_i(s) \frac{1/(R_1 C_1 R_2 C_2)}{\frac{s}{R_2 C_1} + (R_1 C_1 s + 1)\left(\frac{s}{R_1 C_1} + \frac{1}{R_1 C_1 R_2 C_2}\right)} && (14.24)\\
&= V_i(s) \frac{1/(R_1 C_1 R_1 C_2)}{\frac{s}{R_2 C_1} + s^2 + \frac{s}{R_1 C_1} + \frac{s}{R_2 C_2} + \frac{1}{R_1 C_1 R_2 C_2}} && (14.25)\\
&= V_i(s) \frac{1/(R_1 C_1 R_2 C_2)}{s^2 + s\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2}\right) + \frac{1}{R_1 C_1 R_2 C_2}}. && (14.26)
\end{aligned}$$

The transfer function of this filter is the expression in (14.26) to the right of $V_i(s)$. It could also be derived via the node-voltage method; no particular method is necessarily faster of easier than the others.

14.4 Coil Guns

Coil guns cajole electromagnetism into accelerating metal bolts to high speeds. The bolt is placed within the coils of an inductor, near one end. When a current is run through the coils, the bolt is pulled towards the center of the inductor. Precise timing is needed, since the current must be shut off just before the bolt passes through the center of the coil so that it is allowed to leave the coil instead of being pulled back in.

Effective coil guns require large inductor currents. Batteries can store tremendous amounts of electric charge, but they usually cannot provide those charges very quickly; there are practical limits to the amount of current that a battery may provide. A common way to get around this limitation of batteries is to charge a capacitor and use charge from the capacitor to power the coil.

Consider this simplified coil gun model shown in Figure 14.3. V_B represents a battery[1] used to charge the capacitor C up to V_B volts. R_c represents a current limiting resistor introduced to make sure the charging doesn't happen too fast (which might generate excessive heat), and R_p represents parasitic resistances in the coil. Before $t = 0$, the coil is disconnected and the capacitor is charged. After $t = 0$, the charging circuit has been disconnected and the capacitor provides current to the coil.

[1] In practice, the "battery" V_B would likely be provided by a battery combined with some other circuitry to step up the raw battery voltage.

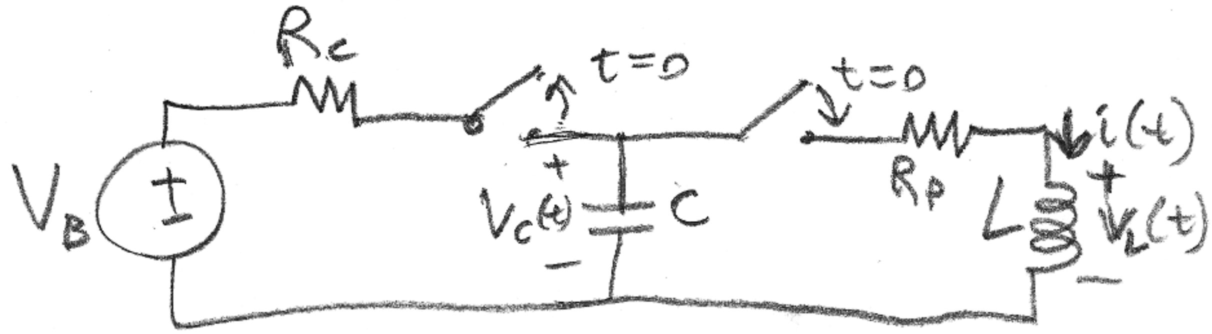


Figure 14.3: Simplified coil gun circuit.

We will assume that $i(0^-) = i(0^+) = 0$, i.e., no energy is stored in the inductor at the beginning; this implies that no voltage is dropping across the resistor at $t = 0$.

[MORE TO COME]

Chapter 15

Feedback

Feedback is an immensely powerful technique used in both natural organisms and engineered systems. In the block diagrams in Figure 15.1, $H_1(s)$ and $H_3(s)$ are *feedforward* systems, and $H_2(s)$ and $H_4(s)$ are *feedback* systems. The tiny minus sign in the left panel denotes negative feedback, which is the focus of most of the discussion of feedback in this book. We can replace the negative feedback structure with a single closed-loop transfer function $H_{CLN}(s)$:

$$Y(s) = H_1(s)[X(s) - H_2(s)Y(s)] \tag{15.1}$$
$$Y(s)[1 + H_1(s)H_2(s)] = H_1(s)X(s) \tag{15.2}$$
$$H_{CLN}(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \tag{15.3}$$

Figure 15.1: Generic linear feedback models: negative feedback (left) and positive feedback (right).

We can replace the positive feedback structure with a single closed-loop transfer function $H_{CLP}(s)$:

$$Y(s) = H_3(s)[X(s) + H_4(s)Y(s)] \tag{15.4}$$
$$Y(s)[1 - H_3(s)H_4(s)] = H_3(s)X(s) \tag{15.5}$$
$$H_{CLP}(s) = \frac{Y(s)}{X(s)} = \frac{H_3(s)}{1 - H_3(s)H_4(s)} \tag{15.6}$$

The distinction between negative and positive feedback may seem somewhat pedantic; mathematically, we could equate them by letting $H_1(s) = H_2(s)$ and $H_3(s) = -H_4(s)$. But, the kinds of engineering problems solved with each kind of feedback, and the mindsets needed to tackle those problems, are so diverse that we feel it is worth emphasizing the difference.

15.1 Amplifier gain/bandwidth tradeoffs

Suppose you were in the market for an amplifier, and found a good deal on the ACME-SFO from the Acme Amplifier Association. Alas, the amplifier isn't perfect; it rolls off high frequencies according to a first-order lowpass response:

$$H_1(s) = \frac{k\omega_c}{s+\omega_c}. \tag{15.7}$$

The half-power cutoff frequency is $\omega_c = 18\pi$ radians/second, corresponding to a rather terrible $6Hz$ cutoff. To get a sense of 6 Hz, imagine a 120 beats per minute song such as "Don't Stop Believin'" by Journey, "Rumour Has It" by Adele, "Bad Romance" or "Poker Face" by Lady GaGa. That will give you two beats per second. Playing triplets within those beats will make you realize that you can drum faster than the ACME-SFO.

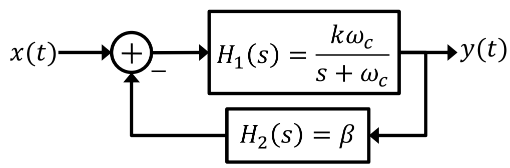


Figure 15.2: Caption goes here

The good news about the ACME-SFO is that it has a massive amount of gain; the gain at DC is $k = 200,000$. We can exploit simple negative feedback (shown in Figure 15.2) with just a scale factor, $H_2(s) = \beta$, to give up some of this gain and improve bandwidth in return.

Using (15.3) reveals the closed-loop transfer function

$$H_{CL}(s) = \frac{\frac{k\omega_c}{s+\omega_c}}{1+\beta\frac{k\omega_c}{s+\omega_c}} = \frac{k\omega_c}{(s+\omega_c)+\beta k\omega_c} = \frac{k\omega_c}{s+(1+\beta k)\omega_c}. \tag{15.8}$$

We have moved the pole from $-\omega_c$ to $-(1+\beta k)\omega_c$, and hence achieved a higher cutoff frequency. This doesn't come for free; the DC gain has been lowered to

$$H_{CL}(j0) = \frac{k}{1+\beta k}. \tag{15.9}$$

Note that as $k \to \infty$, $H_{CL}(j0) \to 1/\beta$. So for extremely large gain k, the closed-loop gain is set by β.

Suppose it turns out that the ACMS-SFO actually has an inverting input, so we can implement negative feedback without having a separate subtraction unit, as shown in Figure 15.3. You may have noticed that we have written the amplifier as a triangle; that's not a coincidence.

The ACME-SFO, is in fact, the common 741 op amp.[1] Now that we've shown what's behind the curtain, let's officially represent the input $x(t)$ and output $y(t)$ as the voltages v_{in} and v_{out}, as shown in Figure 15.4. We are treating the op amp as *almost* ideal: we assume that its output can create whatever voltages and currents are needed, and its inputs draw no current, but we are still assuming it has the transfer function

[1] SFO stood for Seven Forty One. Yes, we were being excessively clever.

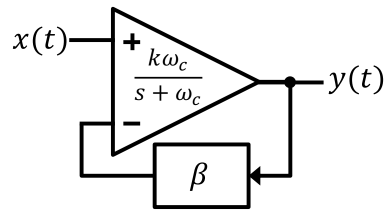


Figure 15.3: Caption goes here

$H_1(s)$, in contrast with typical introductory circuits texts that assume op amps have infinite bandwidth and infinite gain.

We have yet to figure out how to implement β. It's probably best that we keep $\beta < 1$, since having $\beta > 1$ would require another amplifier, which might have the same sort of problems as our original 741, plunging us down a rabbit hole. We can easily set a $\beta < 1$ using a resistive divider, as shown in Figure 15.4, which sets $\beta = R_G/(R_F + R_G)$. For large k, the DC gain is effectively

$$\frac{1}{\beta} = \frac{R_F + R_G}{R_G} = 1 + \frac{R_F}{R_G}. \tag{15.10}$$

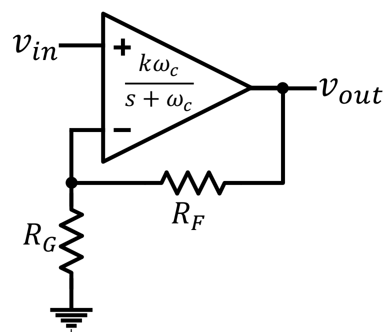


Figure 15.4: Caption goes here

It is extremely convenient that the DC gain can be set by β, with the exact value of k becoming irrelevant as long as it is huge. The $k = 200,000$ value cited earlier is listed as "typical" value on the 741 datasheet, which lists a "minimum" values of $k = 50,000$, and doesn't even bother listing a "maximum" k value. The gain k can vary from unit to unit, and more imporantly, vary drastically with different operating conditions, particularly temperature. In contrast, resistors typically act relatively similar over a wide range of frequencies. This is an example of a general engineering principle: compensate for uncertainties in a system by putting more reliable components in the feedback loop.

The formula in (15.10) should look familiar, particularly if we rearrange this noninverting amplifier configuration as in the form 15.5, which is how this circuit is typically drawn in introductory circuits textbooks. These texts derive (15.10) using Kirchoff's current law after assuming that "golden rules" of an ideal opamp with negative feedback force the voltages at the opamp inputs to be equal. However, such a derivation gives

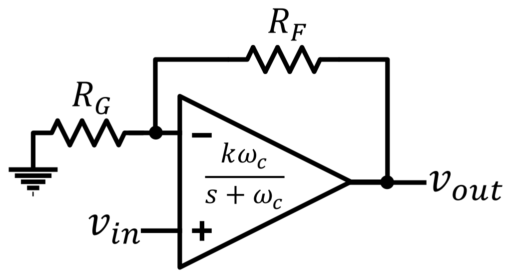
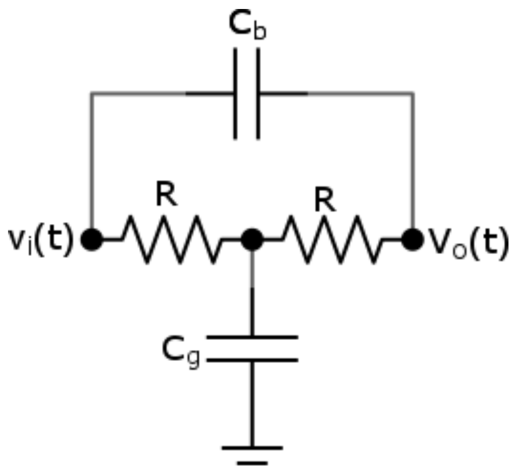


Figure 15.5: Caption goes here

little insight into how someone might come up with the circuit to begin with. The systems-level feedback analysis in this section provides such illumination, since we started with a goal and worked towards it, instead of having a solution provided and analyzing it after the fact.

15.2 Transfer function inversion

The following circuit network is referred to as a *bridged-T* configuration:



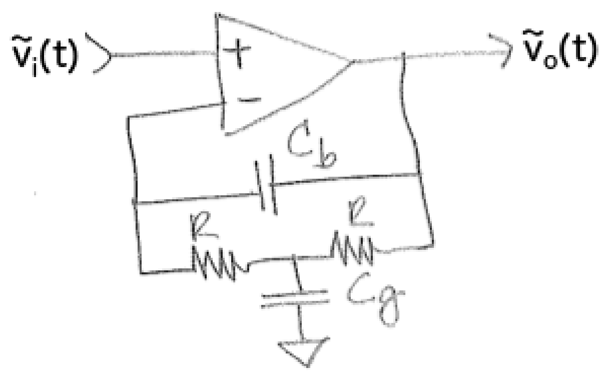
After straightforward but tedious circuit analysis and algebra, the voltage-to-voltage transfer function relating $v_o(t)$ to $v_i(t)$ is found to be:

$$H_{notch}(s) = \frac{V_o(s)}{V_i(s)} = \frac{R^2C_bC_gs^2 + 2RC_bs + 1}{R^2C_bC_gs^2 + R(2C_b + C_g)s + 1}.$$

As usual, we assume that $v_i(t)$ is an ideal voltage source that can generate whatever current it needs, and assume that $v_o(t)$ is measured relative to ground by some ideal device that draws no current.[2]

By adjusting R, C_1, and C_2, notch filters of different frequencies and depths can be designed.

Designing a passive peaking filter using the same bridged-T configuration would require the use of inductors, which tend to be avoided in modern audio applications because they are bulky, expensive, and subject to numerous parasitic effects; resistors and capacitors are relatively small, cheap, and well-behaved in comparison. There are numerous active topologies employing op-amps that could be used to create peaking filters. Here, let's create a peaking filter by putting our bridged-T notch filter in the feedback loop of an op amp. The output of our peaking filter, which is the output of the op amp, is fed to the input of the bridged-T notched filter and the output of the bridged-T notched filter is fed to the negative input of the op amp. The input of the peaking filter is fed to the positive input of the op amp. The schematic looks something like this:



The input and output of the peaking filter are denoted with tildes, as in $\tilde{v}_i(t)$ and $\tilde{v}_o(t)$, to avoid confusion with $v_i(t)$ and $v_o(t)$, which represented input and output of the notch filter in the schematic the previous page.

Remember from your circuits class that an ideal op amp operating with negative feedback produces whatever output current and voltages are needed to make the voltages at its positive and negative input terminals be the same. Hence, the output of the notch filter, $v_o(t)$, equals the input of the peaking filter, $\tilde{v}_i(t)$. Also, the output of the peaking filter, $\tilde{v}_o(t)$, matches the input of the notch filter, $v_i(t)$. This general trick allows us to use an op amp to invert the transfer function of a linear system.[3]

The transfer function of the resulting peaking filter is

$$H_{peak}(s) = \frac{1}{H_{notch}(s)} = \frac{R^2 C_b C_g s^2 + R(2C_b + C_g)s + 1}{R^2 C_b C_g s^2 + 2RC_b s + 1}. \tag{15.11}$$

A bridged-T-in-feedback-loop peaking filter is used in the Sontec MEP-250 Equalizer. Assuming the link is still available, you might enjoy studying this schematic:

`http://gyraf.dk/schematics/Sontec_MEP250a.GIF`

The schematic is rather difficult to follow, so you may not want to study it too much. The basic idea is that this peaking filter can be embedded in a second, larger feedback loop, and the user can switch between peaking and notching behaviors by adjusting a potentiometer. The resistors we called "R" are dual-ganged potentiometers that let the user vary the frequency.

[2] Hence, although the configuration of resistors, capacitors, and inductors appears symmetric, it may be misleading to think of it as symmetric when doing the analysis.

[3] In practice, various effects limit how well this inversion will work.

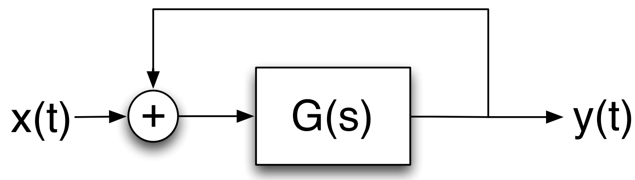


Figure 15.6: Positive feedback.

George Massenburg Labs makes a high-quality modern version. You don't want to know how much it costs:

http://www.massenburg.com/products/gml-8200

Here's a youtube clip of Massenburg equalizing a piano with the GML 8200:

http://www.youtube.com/watch?v=8kBFqHo2z9Q

15.3 Oscillators

Throughout most of our discussions about feedback, we wax philosophical about the power of negative feedback—but we don't want to leave you with the impression that negative feedback is the only useful kind of feedback. There are times, particularly when designing oscillators, that *positive* feedback can be helpful. Hewlett-Packard's first product was a sinusoidal lab oscillator called the HP 200A.[4] The first of these was built in Dave Packard's garage. This section will analyze an extremely simplified interpretation of the classic HP 200A.

For a *generic* $G(s)$ in this *positive* feedback configuration shown in Figure 15.6, the closed-loop transfer function $H(s) = Y(s)/X(s)$ in terms of $G(s)$ is

$$H(s) = \frac{G(s)}{1 - G(s)}.$$

In Figure 15.7, the resistors each have the same resistance R, and the capacitors each have the same capacitance C. The triangle with K inside of it is not an opamp. It represents an ideal voltage amplifier[5] with a gain of K—the output voltage at the point of the triangle is K times the input voltage at the wire connected to the left edge of the triangle. Assume that this magical amplifier has infinite input impedance (so no current flows into it and it does not load down any circuitry feeding it) and it has zero output impedance (so it is not loaded down by the circuitry that follows it). We assume that there are no limitations on output voltage or output current.

The series combination has impedance $R+1/(Cs)$ and the parallel combination has impedance $[R/(Cs)]/[R+1/(Cs)] = R/(RCs+1)$. The transfer function $G(s) = V_o(s)/V_i(s)$ that relates the output voltage, $v_o(t)$, to the input voltage, $v_i(t)$, where the input and output voltages are both referenced to ground, is readily found

[4] If you are feeling particularly brave, you can read the patent at:
http://www.hp.com/hpinfo/abouthp/histnfacts/museum/earlyinstruments/0002/other/0002patent.pdf, although we don't particularly recommend it.

[5] The amplifier in the HP 200A is less magical, consisting of a couple of vacuum tubes, a handful of capacitors and resistors, and a light bulb. Yes, a light bulb. That's part of the brilliance (pun intended) of Bill Hewlett's design. You can think of it as the Flux Capacitor of the HP 200A; if you don't get the reference, go watch the movie "Back to the Future" *right now.*

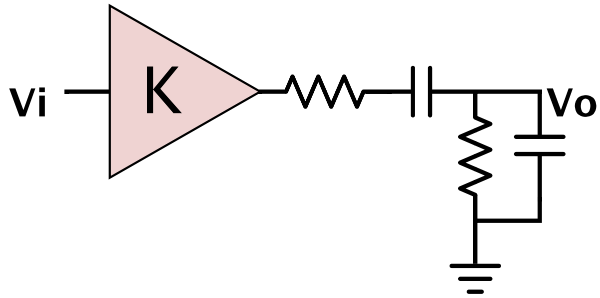


Figure 15.7: Extremely simplified HP200a open-loop structure.

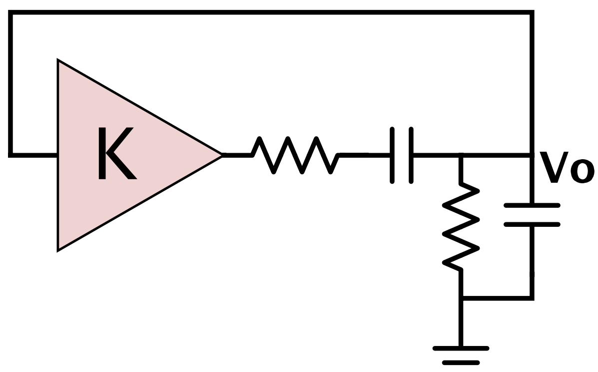


Figure 15.8: Extremely simplified HP200a closed-loop structure.

via voltage division:

$$G(s) = K\frac{\frac{R}{RCs+1}}{\frac{R}{RCs+1} + R + \frac{1}{Cs}} = K\frac{R}{R + [R + 1/(Cs)](RCs+1)} = K\frac{RCs}{RCs + (RCs+1)^2} \quad (15.12)$$

$$= K\frac{RCs}{RCs + R^2C^2s^2 + 2RCs + 1} = K\frac{RCs}{R^2C^2s^2 + 3RCs + 1}. \quad (15.13)$$

Now let's add positive feedback to Figure 15.7. The closed-loop transfer function of Figure 15.8 is derived by plugging (15.13) into (15.3):

$$H(s) = \frac{K\frac{RCs}{R^2C^2s^2+3RCs+1}}{1 - K\frac{RCs}{R^2C^2s^2+3RCs+1}} = \frac{KRCs}{R^2C^2s^2 + 3RCs + 1 - KRCs} = \frac{KRCs}{R^2C^2s^2 + (3-K)RCs + 1}. \quad (15.14)$$

To form an oscillator, we would like the closed-loop system to be marginally stable. Setting $K = 3$ eliminates the middle term in the denominator and place the poles on the imaginary axis at $\pm j\omega_n$, where $\omega_n = 1/RC$.

You might find it a bit strange that there's not really any external input. In practice, internal noise is usually sufficient to jump-start such circuits into oscillation.

Chapter 16

Control Systems

16.1 The trouble with open loop control

Consider a system, which we will call the "plant," with transfer function $G_p(s)$, input $x(t)$, and output $y(t)$. The output of the plant in the s-domain is $Y(s) = X(s)G_p(s)$. We might ask what input $X(s)$ would generate some desired output $Y(s)$. Mathematically, we could write

$$X(s) = \frac{Y(s)}{G_p(s)}. \tag{16.1}$$

As we will see, the innocuous-looking equation (16.1) is fraught with peril.

Consider a plant with a single pole on the negative real axis:

$$G_p(s) = \frac{k}{s+a}. \tag{16.2}$$

If we wanted the output to exactly match a unit step, brute-force application of (16.1) would give

$$X(s) = \frac{1}{s \cdot \frac{k}{s+k}} = \frac{s+a}{sk} = \frac{1}{k} + \frac{a}{ks}. \tag{16.3}$$

Taking the inverse Laplace transform of (16.3) yields

$$x(t) = \frac{1}{k}\delta(t) + \frac{a}{k}u(t). \tag{16.4}$$

We would need to "hit" the system with an impulse to get truly "instantaneous perfect tracking," which is not feasible. Hence, we must back off on our demands. A less stringent goal would be to require the output of the plant to eventually be want we want, but not expect that it will get there right away. To seek the simplest $x(t)$ that might do this, we could consider dropping the problematic impulsive term in (16.4), which corresponds to the constant term in (16.3). The input $x(t) = (a/k)u(t)$ yields the Laplace-domain output

$$Y(s) = X(s)G_p(s) = \frac{a}{ks} \cdot \frac{k}{s+a} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}, \tag{16.5}$$

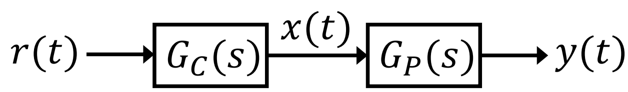


Figure 16.1: Generic open loop control scheme.

with the corresponding time-domain output

$$y(t) = (1 - e^{-at})u(t). \tag{16.6}$$

As $t \to \infty$, the second term decays to zero, so we have achieved our goal of "perfect" steady-state tracking; asymptotically, the output approaches a unity. Because "instantaneous" perfect tracking behavior is never realistic, we will henceforth use the term "tracking" to refer to steady-state tracking performance without modifiers such as "asymptotic."

In the above example, we could think of the multiplication by a/k as a second system, called the *controller*, with system function $G_c(s) = a/k$, that acts on the reference signal $r(t) = u(t)$, to generate $x(t)$, the input to the plant. More generally, the output of such an open-loop control system is specified by $Y(s) = R(s)G_c(s)G_p(s)$, as shown in Figure 16.1.

What if we wanted to speed up or slow down the system response by changing the pole location, say from $s = -a$ to $s = -\alpha$? We might try a controller with the form

$$G_c(s) = \frac{B(s+a)}{s+\alpha}, \tag{16.7}$$

which would cancel out the undesired pole at $s = -a$ and insert the desired pole at $s = -\alpha$.

For a unit step reference signal, $r(t) = u(t)$, the output of the plant is

$$Y(s) = R(s)G_c(s)G_p(s) = \frac{1}{s} \cdot \frac{B(s+a)}{s+\alpha} \cdot \frac{k}{s+a} = \frac{kB}{s(s+\alpha)}. \tag{16.8}$$

As expected, the zero at $s = -a$ of $G_c(s)$ cancels the pole $s = -a$ of $G_p(s)$.

The final value theorem can be applied to (16.8) to obtain the steady-state output,[1]

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \left.\frac{kB}{s+\alpha}\right|_{s=0} = \frac{kB}{\alpha}. \tag{16.9}$$

To get $y_{ss} = 1$, we need $B = \alpha/k$, so the controller becomes

$$G_c(s) = \frac{\alpha(s+a)}{k(s+\alpha)}. \tag{16.10}$$

The plant output is now

$$y(t) = (1 - e^{-\alpha t})u(t). \tag{16.11}$$

[1] This is the traditional approach control theory texts use to find steady-state outputs. One could equivalently treat $G_c(s)G_p(s)$ as a filter and find its DC response, i.e. calculate $G_c(j0)G_p(j0) = G_c(0)G_p(0)$. Notice that in applying the final value theorem, the s from the theorem cancels the s in the denominator arising from the Laplace transform of a unit step, so taking the limit as $s \to 0$ yields an expression equivalent to finding the DC response.

In practice, the "real" plant is rarely exactly characterized by the "model" plant $G_p(s)$. For our example of a plant with a single pole, the value of k, the overall gain of the plant, or a, the pole location, will probably differ from that of the model. Suppose that the transfer function of the actual plant is

$$G_p(s) = \frac{k + \Delta k}{s + a + \Delta a}, \tag{16.12}$$

where both the plant gain and its pole location are not exactly as expected. For a unit step reference input, the output in the s-domain is now

$$Y(s) = \frac{1}{s} \cdot \frac{a(s+a)}{k(s+\alpha)} \cdot \frac{k+\Delta k}{s+a+\Delta a} = \frac{k+\Delta k}{k} \frac{\alpha(s+a)}{s(s+\alpha)(s+a+\Delta a)}. \tag{16.13}$$

The zero in the controller no longer cancels out the pole in the plant. Also, the gain of the controller is no longer the right value to achieve perfect tracking.

The first implication is that the steady state value of the output may no longer be unity. If we assume that the overall system is stable, we can apply the final value theorem:

$$y_{ss} = \lim_{s \to 0} sY(s) = \frac{k+\Delta k}{k} \frac{\alpha a}{\alpha(a+\Delta a)} = \frac{k+\Delta k}{k} \frac{a}{(a+\Delta a)}. \tag{16.14}$$

If $\Delta k = 0$ and $\Delta a = 0$, the plant matches the model and the system perfectly tracks a step. But if the plant deviates from the model, it will not perfectly track, unless you have the extremely unlikely situation where modeling errors in k and a cancel out; i.e., $\Delta k/k = -\Delta a/a$).

The second implication is that the pole at $s = a + \Delta a$ "leaks" through because it is not perfectly canceled out. This can be seen by performing a partial fraction expansion on $Y(s)$:

$$Y(s) = \frac{k+\Delta k}{k} \frac{\alpha(s+a)}{s(s+\alpha)(s+a+\Delta a)} = \frac{c_1}{s} + \frac{c_2}{s+\alpha} + \frac{c_3}{s+a+\Delta a}. \tag{16.15}$$

In the time domain, we have

$$y(t) = c_1 u(t) + c_2 e^{-\alpha t} u(t) + c_3 e^{-(a+\Delta a)t} u(t). \tag{16.16}$$

The third term is caused by the "bad" pole that has not been canceled out. If c_3 is small and $a + \Delta a > 0$ (pole is in the left half plane), then this may not be much of a problem. However, if the plant is unstable, i.e., $a + \Delta a < 0$ (pole is in the right half plane), then it is a disaster!

Let's consider a specific example:

$$G_p(s) = \frac{10}{s-5}. \tag{16.17}$$

This plant is unstable since it has a pole at $s = 5$, which is in the right half plane. Suppose we want the pole to be at $s = -2$, which is in the left half plane. The required controller for our open-loop scheme is

$$G_c(s) = \frac{2(s-5)}{10(s+2)}. \tag{16.18}$$

The transfer function of the resulting open-loop system is

$$H(s) = G_c(s) G_p(s) = \frac{2}{s+2}. \tag{16.19}$$

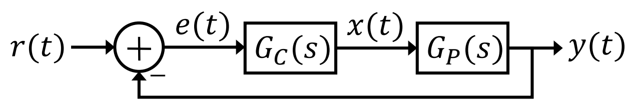


Figure 16.2: Generic feedback control scheme.

But now suppose that the transfer function of the "real" plant does not match that of the model, but is instead

$$G_p(s) = \frac{10}{s - 5.1}. \tag{16.20}$$

This is a small deviation, but it causes big problems! The transfer function of the open loop system now becomes

$$H(s) = G_c(s)G_p(s) = \frac{2(s-5)}{10(s+2)} \cdot \frac{10}{s-5.1} = \frac{2(s-5)}{(s+2)(s-5.1)}. \tag{16.21}$$

This system is unstable because the zero in the controller does not cancel out the pole in the plant. The response to a step function now becomes

$$y(t) = c_1 u(t) + c_2 e^{-2t} u(t) + c_3 e^{5.1t} u(t). \tag{16.22}$$

Even if c_3 is extremely small, the growing exponential term will still eventually dominate and the system output will "blow up."

Although open-loop control may be satisfactory for some applications, it generally cannot accommodate small changes in the plant that as are likely to occur over time, and may even result in instability.

16.2 The general setup for feedback control

The generic feedback control scheme illustrated in Figure 16.2 has the closed-loop system function

$$H(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}. \tag{16.23}$$

16.3 P control

Consider a plant with a single pole on the real axis:

$$G_p(s) = \frac{k}{s+a}. \tag{16.24}$$

A simple and intuitive approach might be to let the input to the plant be proportional to the error signal, i.e., choose $G_c(s) = K_p$. The resulting closed-loop system function is

$$H(s) = \frac{\frac{kK_p}{s+a}}{1 + \frac{kK_p}{s+a}} = \frac{kK_p}{s + a + kK_p}. \tag{16.25}$$

Notice that feedback allowed us to move the pole from $-a$ to $-(a + kK_p)$. If $a \leq 0$, then the plant itself is unstable, and choosing K_p so that $a + kK_p > 0$ performs the amazing feat of stabilizing the system. We might want to apply feedback control even if the plant was stable to begin with; moving the pole to the left in the s-plane allows the closed-loop system to respond "faster" than the original open-loop system. Although one might think that "faster" is always "better," this might depend on the application. For instance, an elevator that started or stopped too suddenly might frighten, or worse, even injure its passengers!

For a unit-step reference input, the output is $Y(s) = H(s)/s$, and the final value theorem yields the yields the steady-state output

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{H(s)}{s} = H(s)|_{s=0} = \frac{kK_p}{a + kK_p}, \tag{16.26}$$

assuming $a + kK_p > 0$ so that $H(s)$ is BIBO stable. One could also view $H(s)$ as a filter and simplify find the DC response $H(j0)$ (again, assuming $H(s)$ is BIBO stable). Although we would like y_{ss} to be 1 for a *unit*-step reference input, and we see that y_{ss} approaches one as K_p is increased, we see that it never quite "gets there." Although one might be tempted to simply crank K_p until the tracking performance is "good enough," there are often practical physical limits – voltage swings or digital word lengths in a control system implementation, motor speed, heating coil temperature, etc. – that prevent us from turning K_p as high as we might like.

16.4 PI control

Consider a plant with one pole at the origin and another elsewhere along the real axis:

$$G_p(s) = \frac{1}{s(s+a)}. \tag{16.27}$$

Let us try simple proportional control, i.e., $G_c(s) = K_p$. The closed-loop system function is

$$H(s) = \frac{\frac{K_p}{s(s+a)}}{1 + \frac{K_p}{s(s+a)}} = \frac{K_p}{s(s+a) + K_p} = \frac{K_p}{s^2 + sa + K_p}. \tag{16.28}$$

If the reference input is a unit step, $r(t) = u(t)$, the Laplace transform of the output is is given by

$$Y(s) = \frac{K_p}{s(s^2 + sa + K_p)}. \tag{16.29}$$

Assuming the poles of (16.28) are in the left half of the s-plane, so that $H(s)$ is stable, the final value theorem – or equivalently, the DC frequency response $H(j0)$ – yields the steady-state output

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{H(s)}{s} = H(s)|_{s=0} = \frac{K_p}{K_p} = 1. \tag{16.30}$$

So, unlike in the case explored in Section 16.3 where the plant had just a single pole on the real axis, the same kind of system with the additional pole at the origin has zero tracking error.[2] If the plant lacks a pole at the origin, we might be inspired to try achieving zero tracking error by introducing a pole at the origin *in the controller*:

$$G_c(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}. \tag{16.31}$$

[2]Remember, statements about tracking implicit include a modifier such as "steady-state."

This is referred to as a Proportional-Integral (PI) controller.

The input to the plant, in the Laplace domain, is

$$X(s) = E(s)G_c(s) = K_pE(s) + \frac{K_i}{s}E(s),$$

where $E(s)$ specifies the error signal. Since division by s in the Laplace domain corresponds to integration in the time domain, the input to the plant is

$$x(t) = K_pe(t) + K_i\int_0^t e(\tau)d\tau.$$

Let us once again consider a plant with a single pole on the real axis, not at the origin:

$$G_p(s) = \frac{k}{s+a}. \tag{16.32}$$

Applying a PI controller yields the closed-loop system function

$$H(s) = \frac{\frac{K_ps+K_i}{s}\cdot\frac{k}{s+a}}{1+\frac{K_ps+K_i}{s}\cdot\frac{k}{s+a}} = \frac{k(K_ps+K_i)}{s(s+a)+k(K_ps+K_i)} = \frac{k(K_ps+K_i)}{s^2+(a+kK_p)s+kK_i}. \tag{16.33}$$

Our usual application of the final value theorem, or equivalently, consideration of the DC frequency response $H(j0)$, shows that for a unit step input $r(t) = u(t)$, the steady-state output is 1, i.e., we have perfect tracking of a step function. Interpreting $H(s)$ in the frequency domain, we see that it appears like a combination of second-order lowpass and bandpass responses. This viewpoint makes it obvious that it would not make sense to have a pure "I" controller with $K_p = 0$, since the bandpass response alone would output 0 at "DC."

Assuming we "know" a, two approaches to choosing K_p and K_i might come to mind:

- **A "safe" strategy – move the poles to where we want them:** For instance, one might desire $H(s)$ to exhibit a particular natural frequency ω_n and damping factor ζ. Matching the coefficients of the polynomial in the denominator of (16.33) with our canonical polynomial $s^2 + 2\zeta\omega_ns + \omega_n^2$, we have the equations $a + kK_p = 2\zeta\omega_n$ and $kK_i = \omega_n^2$, yielding the design equations

$$K_p = \frac{2\zeta\omega_n - a}{k},\ K_i = \frac{\omega_n^2}{k}. \tag{16.34}$$

- **A "risky" strategy – cancel one pole and insert another one:** Theoretically, if the plant has a single pole, this approach could yield $H(s)$ with a single pole instead of two poles as in the "safe" strategy. But... ETC. ETC. – [WE'LL FINISH THIS WRITEUP SOME OTHER DAY]

16.5 PD control

PD control employes a controller with the form $G_c(s) = K_p + K_ds$. For general PD control, the input to the plant is expressed in the s-domain as

$$X(s) = E(s)G_c(s) = E(s)(K_p + K_ds). \tag{16.35}$$

In the time domain, this corresponds to

$$x(t) = K_p e(t) + K_d \frac{de(t)}{dt}. \tag{16.36}$$

In cases where the output $y(t)$ represents a position, for instance, of a vehicle on a rail, $\dot{e}(t)$ represents error in velocity. One might roughly think of adjusting K_p and K_d as trading off how the controller prioritizes errors in position and velocity, respectively. (Similarly, if $y(t)$ represented a *velocity* – for instance, in a cruise control system – $\dot{e}(t)$ would represent an error in *acceleration*.)

16.5.1 PD control of a system with two real poles

Consider a plant with one pole at the origin and another elsewhere along the real axis:

$$G_p(s) = \frac{1}{s(s+a)}. \tag{16.37}$$

Such systems are common. For instance, consider a wheeled vehicle in which $y(t)$ represents position. The equation governing the motion of the vehicle is $m\ddot{y} = -\mu\dot{y} + f(t)$, where $x(t)$ is the applied force. The resulting system function is

$$\frac{1/m}{s(s+d/m)}, \tag{16.38}$$

which matches (16.37) with $a = d/m$ and $k = 1/m$.

Notice the system (16.37) is not BIBO stable. This is not surprising; for instance, if we go into a grocery store, grab a shopping cart, and start pushing it with a constant force, we will keep going until we break assumptions of system linearity by encountering a wall or another (not probably quite irate) customer.

Trying $G_c(s) = K_p + K_d s$ yields the closed-loop system function

$$H(s) = \frac{(K_p + K_d s)\frac{k}{s(s+a)}}{1 + (K_p + K_d s)\frac{k}{s(s+a)}} = \frac{k(K_p + K_d s)}{s(s+a) + k(K_p + K_d s)} = \frac{k(K_p + K_d s)}{s^2 + (a + kK_d)s + kK_p}. \tag{16.39}$$

Its DC gain is $H(j0) = K_p/K_p = 1$, so it can perfectly track a step function. Equating the denominator of (16.39) with our canonical denominator $s^2 + 2\zeta\omega_n s + \omega_n^2$ yields $a + kK_d = 2\zeta\omega_n$ and $kK_p = \omega_n^2$, giving the following design equations for pole placement:

$$K_p = \frac{\omega_n^2}{k},\ K_d = \frac{2\zeta\omega_n - a}{k}. \tag{16.40}$$

16.5.2 "D" stands for—Danger???

Taking derivatives can be dicey, whether the derivative operation is implemented, for instance, by an operational amplifier with capacitor in its local negative feedback path, or digitally via a micro controller. The frequency response of the derivative operation is $H(j\omega)$; this is not just a highpass filter—it is a license for high frequencies to go wild. Noise – which we have generally ignored, but which is important to think about in real-world applications – tends to be heavily amplified by a derivative operation. Hence, in practice, the input to the "D" part of the PID controller is typically first passed through some kind of filter to reduce the level of noise. (This is not an issue with the "I" part of the controller, since the integration operation inherently tries to "average out" noise).

16.6 Tracking inputs that are not step functions

16.6.1 Tracking sinusoids

Suppose we wanted to track a sinusoidal reference signal $r(t) = \cos(\omega_0 t)u(t)$. Its Laplace transform is

$$R(s) = \frac{s}{s^2 + \omega_0^2}. \tag{16.41}$$

Since the desired output oscillates eternally – the desired steady-state output, $y_{ss}(t)$, is no longer a constant as in our step-input examples – the final value theorem is of less use here. We could consider two approaches to analyzing the tracking characteristics of a control system with closed-loop system function $H(s)$:

- Applying partial fraction expansion to the output yields the general form

  $$Y(s) = R(s)H(s) = \frac{c}{s - j\omega_0} + \frac{c^*}{s + j\omega_0} + \text{other terms}. \tag{16.42}$$

  Assuming the closed-loop system is BIBO stable, the inverse Laplace transforms of the "other terms" asymptotically trend towards zero, so the steady-state output is

  $$y_{ss}(t) = 2|c|\cos(\omega_0 + \angle\{c\}).$$

- We could equivalently compute the frequency response of the closed-loop system, $H(j\omega)$, plug the reference frequency into that frequency response, and use the notion of "sinusoid in" leading to "sinusoid out" to find

  $$y_{ss}(t) = 2H(j\omega)\cos(\omega_0 + \angle\{H(j\omega)\}).$$

16.6.2 Tracking ramps

Imagine we wanted to track a "unit ramp," i.e., $r(t) = tu(t)$, instead of a unit step. The Laplace-domain representation of this ramp is $R(s) = 1/s^2$. Applying partial fraction expansion to the output yields the general form

$$Y(s) = R(s)H(s) = \frac{c_1}{s} + \frac{c_2}{s^2} + \text{other terms}. \tag{16.43}$$

If the closed-loop system is BIBO stable, the inverse Laplace transforms of the "other terms" go to zero over time, and the steady-state output is

$$y_{ss}(t) = c_1 u(t) + c_2 t u(t). \tag{16.44}$$

We have perfect tracking of the reference ramp if $c_1 = 0$ and $c_2 = 1$. If $c_2 \neq 0$, then the absolute error will grow with time. Having $c_2 = 1$ but $c_1 \neq 0$ corresponds to a constant "bias" error.

16.7 PID control of a resonant system

Suppose the plant has the system function

$$G_p(s) = \frac{k}{s^2 + \omega_0^2}. \tag{16.45}$$

This system is not BIBO stable. It exhibits a resonance at ω_0.

Suppose we would like to stabilize this system and make it perfectly track a stack a step; of course, achieving the former is necessary before we can even think about doing the latter. Let us try the various controllers in our toolbox, starting with the simplest:

- **P control:** With $G_c(s) = K_p$, the closed-loop system function is

$$H(s) = \frac{\frac{kK_p}{s^2+\omega_0^2}}{1 + \frac{kK_p}{s^2+\omega_0^2}} = \frac{kK_p}{s^2 + \omega_0^2 + kK_p}. \tag{16.46}$$

  We can forget about tracking with a P-controller, since it cannot even stabilize the system. All it does it shift the resonance from ω_0 to $\sqrt{\omega_0^2 + kK_p}$.

- **PI control:** With $G_c(s) = K_p + K_i/s$, the closed-loop system function is

$$H(s) = \frac{\frac{K_p s + K_i}{s} \cdot \frac{k}{s^2+\omega_0^2}}{1 + \frac{K_p s + K_i}{s} \cdot \frac{k}{s^2+\omega_0^2}} = \frac{k(K_p s + K_i)}{s^3 + (kK_p + \omega_0^2)s + kK_i}. \tag{16.47}$$

  Here, we wind up three poles in the closed-loop system function. It turns out that the missing s^2 term in the denominator is problematic; it implies that there is a conjugate pair of poles on the imaginary axis.[3] So a PI controller cannot stabilize the system either.

- **PD control:** With $G_c(s) = K_p + K_d s$, the closed-loop system function is

$$H(s) = \frac{(K_d s + K_p)\frac{k}{s^2+\omega_0^2}}{1 + (K_d s + K_p)\frac{k}{s^2+\omega_0^2}} = \frac{k(K_d s + K_p)}{s^2 + kK_d s + kK_p + \omega_0^2}. \tag{16.48}$$

  Now we are getting somewhere. Unlike in the case of PI control, we have two poles instead of three, and these two poles can be controlled with the two "knobs" K_p and K_d, so we can stabilize the system. If we wish the denominator to exhibit a certain natural frequency ω_n and damping factor ζ, we can again use our coefficient-matching trick, yielding $2\zeta\omega_n^2 = kK_d$ and $\omega_n^2 = kK_p + \omega_0^2$. This gives us the design equations

$$K_p = \frac{\omega_n^2 - \omega_0^2}{k},\ K_d = \frac{2\zeta\omega_n}{k}. \tag{16.49}$$

  Although our ability to relocate the poles allows us to stabilize the system, our task is not yet complete, since this controller cannot perfectly track a step. The DC value of the frequency response of (16.48) is

$$H(j0) = \frac{kK_p}{kK_p + \omega_0^2}. \tag{16.50}$$

  The observations made at the end of Section 16.3 apply here as well; although we might theoretically want to to increase K_p to "swamp" ω_0^2 and get $H(j0)$ closer to 1, real-world considerations limit how far we can push K_p.

[3] This fact and others along these lines are covered thoroughly in most dedicated control theory textbooks; for our current purposes, you can take this on faith.

- **PID control:** With $G_c(s) = K_p + K_i/s + K_d s$, the closed-loop system function is

$$H(s) = \frac{\frac{K_d s^2 + K_p s + K_i}{s} \cdot \frac{k}{s^2+\omega_0^2}}{1 + \frac{K_d s^2 + K_p s + K_i}{s} \cdot \frac{k}{s^2+\omega_0^2}} = \frac{k(K_p s + K_i)}{s^3 + kK_d s^2 + (kK_p + \omega_0^2)s + kK_i} \tag{16.51}$$

  The three denominator coefficients can be set with the three "knobs" K_p, K_i, and K_d, allowing us to place the three poles wherever we want. The PID controller also permits perfect tracking of a step function, since tis DC gain is $H(j0) = 1$.

16.7.1 Example

Consider the plant system function

$$G_p(s) = \frac{1}{s^2+9}, \tag{16.52}$$

which corresponds to (16.45) with $k = 1$ and $\omega_0 = 3$. Suppose we wanted to use PD control to place both closed-loop system poles at $s = -4$; in this case, we would like the denominator of the closed-loop transfer function to be $(s+4)^2 = s^2 + 8s + 16$. Matching with the denominator of (16.48) yields $K_d = 8$ and $K_p = 16 - 9 = 7$. Our satisfaction at our PD controller being able to relocate our poles is short-lived once we consider how terrible it is at tracking a unit step function; from (16.50), $H(j0) = 7/(7+9) = 7/15 \approx 0.4375$, which is dreadfully less than 1.

To apply PID in search of better tracking, we need to make a decision about the third pole. For instance, we might place it at $s = -10$, corresponding to a "fast" response, so that the double pole at $s = -4$ will largely dominate the system response. We want the denominator of (16.51) to match $(s+10)(s+4)^2 = s^3 + 18s^2 + 96s + 160$, which yields $K_d = 18$, $K_p = 96 - 9 = 87$, and $K_i = 160$. The controller has the system function

$$G_c(s) = 87 + \frac{160}{s} + 18s. \tag{16.53}$$

We know from the analysis in Section 16.7 that the complete system perfectly tracks a step. If we are interested in further details of the closed-loop system's unit-step response, a partial fraction expansion yields

$$Y(s) = \frac{H(s)}{s} = \frac{18s^2 + 87s + 160}{s(s^3 + 18s^2 + 96s + 160)} = \frac{1}{s} - \frac{3.028}{s+10} + \frac{2.2028}{s+4} - \frac{4.167}{(s+4)^2}. \tag{16.54}$$

Taking the inverse Laplace transform of Y(s) gives

$$y(t) = (1 - 0.3028e^{-10t} + 2.2028e^{-4t} - 4.167te^{-4t})u(t). \tag{16.55}$$

16.8 PID control in the real world

Scholarly textbooks and papers contain myriad strategies for adjusting the parameters of a PID controller. Engineers who actually implement real PID control systems for a living, instead of just sitting behind a desk thinking about them, typically follow a procedure along these lines:

1. Set K_p, K_i, and K_d to zero.
2. Crank K_p until you observe oscillations in the output, then back off a little.

3. Crank K_i until you observe oscillations in the output, then back off a little.
4. Tweak K_d to reduce overshoots. K_d is like spice in a recipe; a little goes a long away. Seasoning a PID controller with a little bit of "D" sometimes will let you turn up K_p and K_i beyond what you might otherwise get away with.
5. Go back to step (2) and continue twiddling until you achieve the performance you want or you get bored and give up.

Chapter 17

Energy and Power

17.1 Parseval's theorem

Parseval's theorem, in the context of Fourier transforms, tells us that the energy in one domain is proportional to the energy in the other domain:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega. \tag{17.1}$$

Some engineering texts neglect to mention that for the theorem to hold, $x(t)$ and $X(j\omega)$ must be "square integrable," i.e. the integrals on both sides of (17.1) must be finite. For instance, recall that the Fourier transform of $x(t) = \cos(t)$ is $X(j\omega) = \pi\delta(t+1) + \pi\delta(t-1)$. The LHS of (17.1) for this example is infinity. Unfortunately, the RHS teeters between sense and nonsense, ultimately tipping in the direction of nonsense, since the "square of a Dirac delta" cannot be given a definition that is consisting with the properties one might want such mathematical curiosity to have.[1]

There is a variation of Parseval's theorem that applies to periodic functions. Nontrivial periodic functions have infinite energy, so this version integrates over just one period:

$$\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2, \tag{17.2}$$

where we can take the integral over any period of length T_0, and for the theorem to hold, both sides of (17.2) must be finite. For instance, it would not make sense to apply Parseval's theorem to an impulse train, since the RHS would be infinite, which at least is mathematically defined, unlike the LHS, for which we must throw up our hands at the prospect of taking the square of a Dirac delta function. Remember that although Dirac delta "functions" are extremely useful, they contain subtle dangers.

A note on names: Picky mathematical historians may object that "Parseval's theorem" originally referred to the Fourier series version (17.2), and the more general Fourier transform version we started with (17.1) would be more accurately called "Plancherel's theorem." But in science and engineering, we generally just call them all "Parseval's theorems."

[1]Remember that $\delta(t)$ is not an ordinary function, and any expression containing it is not an ordinary function either; trying to treat them as such can get you into all sorts of trouble.

17.1.1 Generalizations for inner products

There are generalizations of (17.1) and (17.2) that treat inner products. Inner products involve integrating (or summing) one function multiplied by the complex conjugate of the other. We touched on the notion of inner products when we discussed Fourier series, and found that a vital part of how Fourier series "work" is that harmonic sinusoids are orthogonal, meaning that the inner products of harmonic sinusoids are zero unless they have the same frequency.

These generalizations of Parseval's theorem tell us that inner products in one domain are proportional to inner products in the other domain.

For square-integrable (i.e. finite energy) functions $f(t)$ and $g(t)$,

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G^*(j\omega)d\omega. \tag{17.3}$$

To derive this, we begin with the multiplication property (8.11) of Fourier transforms,

$$f(t)g(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi}F(j\omega) * G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\xi)G(j(\omega-\xi))d\xi \tag{17.4}$$

and apply the mirror property to $g(t)$, yielding

$$f(t)g^*(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\xi)G^*(j(\xi-\omega))d\xi. \tag{17.5}$$

Explicitly writing the Fourier transform operation yields

$$\int_{-\infty}^{\infty} f(t)g^*(t)\exp(-j\omega t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\xi)G^*(j(\xi-\omega))d\xi. \tag{17.6}$$

Setting $\omega = 0$ in (17.6) and changing the ξ on the right hand side to a "new" ω reveals (17.3). Setting $g(t) = f(t)$ in (17.3) immediately yields (17.1).

For periodic functions $f(t)$ and $g(t)$ with the same period T_0, if a_k and b_k are the Fourier series coefficients of the periodic functions, then

$$\frac{1}{T_0}\int_{T_0} f(t)g^*(t)dt = \sum_{k=-\infty}^{\infty} a_k b_k^* \tag{17.7}$$

if $f(t)$ and $g(t)$ are both square integrable[2] over one period. Setting $g(t) = f(t)$ in (17.7) immediately yields (17.2).

17.2 Power supply design example—guitar amplifiers

17.2.1 Hungry, hungry amplifiers

Before transistors were invented, vacuum tubes – called "valves" by our friends across the pond – dominated the electronics scene. They would have gone the way of the dinosaur by now if it had not been for one highly

[2] Mathematicians can conjure many examples of periodic functions that are not square integrable, but practicing engineers rarely encounter such monstrosities.

Figure 17.1: One day, there might be a figure here. There might even be a caption.

Figure 17.2: One day, there might be a caption and figure here too.

devoted target market: guitarists. When overdriven, tubes are said to distort in a much more musically pleasing way than transistors.[3] We should note that most amplifier designers in the 50s and 60s went to great lengths to *prevent* their amplifiers from distorting, and considered it a badge of shame when they did, even as rock musicians searched for different ways to get their amps to distort *more*.

The power supply is an important part of tube amp design; tubes are power-hungry, voltage craving creatures. Some such circuits operate at voltage levels that can be lethal; even though you are currently just reading these words, you should still feel a bit of fear. It is good practice.

17.2.2 Mighty, mighty Bassman

We will explore a highly simplified, approximated version of the power supply for the Fender Bassman 5F6-A, which is a common ancestor of many modern tube amps. It turned out to be more popular with guitarists than it was with bassists. Jim Marshall, whose amps are associated with acts like Led Zeppelin, AC/DC, and Van Halen, got his start by making amps that were not much more than slightly tweaked versions of the Fender Bassman.

We will start our model with the 120 V AC RMS that comes out of the wall. This is run through a center-tapped transformer that steps *up* the voltage (unlike the transformers in most power supplies people deal with nowadays, which step the voltage down *down*). This is then run through a dual rectifier tube—these tubes act like the solid state diodes you are used to, except they are much scarier. In our model, we will say this setup provides a full-wave rectified sinusoidal wave that goes from 0 volts to $650\sqrt{2}$ volts. The wall current has a frequency of 60 Hz, so the rectified version has a frequency of 120 Hz, with half the period, as shown in Figure 17.1.

We want to filter this rectified sinusoid such that only DC, i.e. the "zero" frequency, gets through. We will never able able to do this perfectly and must make some compromises. The part of the signal that is not constant is called "ripple," and we can characterize it by both its average power and its peak-to-peak swing. Parseval's theorem will give us an easy way to characterize its average power. On a practical note, there is usually a tradeoff between ripple reduction and circuit complexity (and hence size and cost), and another tradeoff between ripple reduction and available DC out—you could always reduce what you do not want if you are willing to lose what you do want!

The Bassman uses an LC lowpass filter consisting of a 10 H inductor in series with a 20 μF capacitor. The output of the rectifier connects to one side to the inductor, one side of the capacitor goes to ground, and the power supply output is at the junction of the inductor and the capacitor, as shown in Figure 17.2.

Power supply ripple can be a nightmare in audio amplifiers, since the ripple will work its way into every audio processing stage. If you have a guitar amp that sounds like it is taking meditation classes, checking the power supply capacitors is a good place to start debugging.

To derive the frequency response of the filter, recall that the impedance of a capacitor is $1/(j\omega C)$ and

[3] Cue endless internet flame wars over why.

the impedance of an inductor is $j\omega L$. Using the voltage divider rule, the frequency response is

$$H(j\omega) = \frac{1/(j\omega C)}{1/(j\omega C) + j\omega L} = \frac{1}{1 + j\omega L(j\omega C)} = \frac{1}{1 - \omega^2 LC}. \quad (17.8)$$

A rectified cosine wave (where the original cosine wave has amplitude 1) has Fourier series coefficients given by (we will not slog through the whole derivation here)

$$a_k = \frac{2}{\pi(1 - 4k^2)}. \quad (17.9)$$

Using linearity, the Fourier series coefficients of the waveform input are just $(650\sqrt{2})a_k$.

The Fourier series coefficients for the output of the filter are

$$b_k = (650\sqrt{2})a_k H(j240\pi k) = 650\sqrt{2}\frac{2}{\pi(1 - 4k^2)} \cdot \frac{1}{1 - (240\pi k)^2 LC}. \quad (17.10)$$

What DC voltage will get from our supply, using these approximations? Well, $H(j0) = 1$, so the DC output is $650\sqrt{2}b_0 = 650\sqrt{2} \times 2/\pi = 585$ volts. That is a lot of volts! Put one hand behind your back before poking around—and remember capacitors can store massive charges long after you have switched off the amp.

The amplitude of the fundamental is

$$2(b_1) = 2(650\sqrt{2})\left(-\frac{2}{3\pi}\right)(-0.0089) = 2(1.74) = 3.48 \quad (17.11)$$

The amplitudes of the second and third harmonics are

$$2(b_2) = (650\sqrt{2})\left(-\frac{2}{15\pi}\right)(-0.0022) = 2(0.0858) = 0.1716 \quad (17.12)$$

and

$$2(b_3) = (650\sqrt{2})\left(-\frac{2}{35\pi}\right)(-.0009) = 2(0.015) = 0.03. \quad (17.13)$$

We have looked at the amplitudes of some individual harmonics. But what is the overall RMS value of the ripple? One could perform an integral like this:

$$\frac{1}{T_0}\left|\int_0^{T_0} \sum_{k\neq 0} b_k \exp(jk240\pi t)\right|^2 dt. \quad (17.14)$$

where $T_0 = 1/120$ seconds, and we have deliberately left the DC value out of the Fourier series sum. But that would be a mess to compute directly? Instead, we can use Parseval's theorem:

$$\sum_{k\neq 0} |b_k|^2. \quad (17.15)$$

Let us add up the first three harmonics. We need to include both the positive and negative coefficients, hence the multiplication by 2 in this expression:

$$2[(1.74)^2 + (0.0858)^2 + (0.015)^2] = 2(3.0267 + 0.0074 + 0.0002) = 6.0686. \quad (17.16)$$

Figure 17.3: A simple circuit with a capacitor, a voltage source, and a switch

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The RMS value is $\sqrt{6.0686} = 2.4635$. Note that the first harmonic by far the strongest contributor.

There is something a little strange about our LC filter that deserves comment. It overall acts as a lowpass filter—clearly the frequency response drops to zero as the frequency becomes clearly large. But it has a"bump" in the response at $\omega_r = 1/\sqrt{LC}$. An *infinitely* big bump, formed by the resonance of the inductor with the capacitor. For our part values, the bump is at $f_r = 1/(2\pi\sqrt{LC}) = 11.25$ Hz, which is not anywhere near the 120 Hz we are worried about removing. Also, the real circuit is no where near this high "quality"—the inductor, in particular, has significant parasitic resistance. So we do not need to worry about 11.25 Hz spontaneously appearing in our amp from thermal noise hitting a resonance by coincidental and getting out of control.

There are a few issues we did not include in our model:

- There is a third capacitor in the power supply, between the output of the rectifier tube and the LC filter listed above, that provides some additional filtering.
- The power supply feeds an additional series of filtering stages, whose outputs power various stages of the amplifier. The push-pull power amp, which by its symmetric design tends to cancel out AC ripple from the power supply, is "closest" to the power supply. The initial preamplifier circuit the guitar sees is "furthest" from the power supply, so its power rail has undergone substantial additional filtering.

A few additional notes:

- Power supply inductors – often called "chokes" – relatively rare nowadays. Most power supplies use a resistor in place of the inductor, forming a one-pole RC filter instead of a two-pole LC filter. A two-pole filter provides a steeper cutoff, especially if you are willing to have a resonance near the cutoff. But chokes are bulky, heavy, expensive, and chock full of non-idealities, which makes the resistor alternative attractive.
- Earlier, we computed a DC output of 585 volts. But if you look on the Bassman schematic, you will see a diagnostic value of 432 volts. Remember that we computed all the numbers above assuming that there was no load on the power supply. The load of the rest of the circuit runs in parallel with the filter capacitor, and cause the voltage at point that to drop. A more detailed analysis would include this load impedance. Also, the non-ideal rectifier tubes, the resistance of the choke, and other factors we have not modeled will result in some loss.

17.3 The Capacitor Paradox

Even as recent as 2015, deep thinkers write and publish new papers on the "capacitor paradox," which was discovered many decades ago. This is usually presented in the form of two capacitors connected in series by a switch. We consider a variation of the paradox consisting of a voltage source with a single capacitor, connected by a switch, as shown in Figure 17.3. This section is inspired by, and closely follows, Problem 11.21 on pp. 361-363 of Siebert.

Suppose the capacitor is uncharged at $t = 0$, and the voltage source has constant voltage V_0. The switch is closed at $t = 0$, so the voltage across the capacitor is $v_c(t) = V_0 u(t)$, and the current through the loop, $i(t)$,

Figure 17.4: Resolving the paradox by adding parasitic resistance.

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which is the same as the current through the capacitor, is $i(t) = C\frac{dv_c(t)}{dt} = CV_0\delta(t)$. The energy delivered by the battery[4] is

$$\int_{-\infty}^{\infty} v_c(t)i(t)dt = \int_{-\infty}^{\infty} [V_0 u(t)][CV_0\delta(t)]dt = \int_{-\infty}^{\infty} V_0^2 C u(t)\delta(t)dt = CV_o^2. \tag{17.17}$$

The energy in the capacitor after the switch is flipped is $CV_0^2/2$. But energy in circuits is supposed to be preserved, so one might ask: where did the other half of the energy go? At first glance, this loss of energy seems to be a “paradox,” but the truly paradoxical nature of this scenario runs much deeper.

17.3.1 One Solution: Add Resistance

Any realistic voltage source, such as a chemical battery, will have some internal resistance, as will any wires connecting the voltages source to the capacitor. Any realistic capacitor will also have some parasitic resistance. We could lump all of these into a single resistance R, seen in Figure 17.4.

The standard solution for a charging capacitor comes into play:

$$v_c(t) = V_0 \left[1 - \exp\left(-\frac{t}{RC}\right)\right] u(t).$$

The current through the loop is

$$i(t) = C\frac{dv_c(t)}{dt} = C\frac{V_o \exp(-t/(RC))}{RC}u(t) = \frac{V_o \exp(-t/(RC))}{R}u(t).$$

The energy delivered by the battery is

$$\int_{-\infty}^{\infty} v_c(t)i(t)dt = \int_{-\infty}^{\infty} [V_0 u(t)]i(t)dt = \int_{-\infty}^{\infty} \frac{V_0^2}{R}\exp(-t/(RC))u(t)dt = \left.\frac{V_o^2}{R}\frac{\exp(-t/(RC))}{-1/(RC)}\right|_{t=0}^{t=\infty} = V_0^2 C. \tag{17.18}$$

Note that the energy supplied by the voltage source is the same as that found in (17.17).

The energy in the capacitor after it is charged is still $CV_0/2$. But now we see where the other half of the energy went; it was dissipated in the resistor:

$$\int_{-\infty}^{\infty} i^2(t)Rdt = \int_{-\infty}^{\infty} \frac{V_o^2 \exp(-2t/(RC))}{R^2}Ru(t)dt = \left.\frac{V_o^2}{R}\frac{\exp(-2t/(RC))}{-2/(RC)}\right|_{t=0}^{t=\infty} = \frac{CV_0^2}{2}. \tag{17.19}$$

Interestingly, the energy delivered by the voltage source (17.22) and the energy dissipated by the resistor are independent of R.

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[4]Something is dodgy about this computation; we will return to this issue later.

17.3.2 Another Solution: A Slower Switch

Here, we suppose that switch cannot be activated instantaneously, and model the combination of the voltage source and the imperfect switch as providing a voltage to the capacitor of

$$v_c(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ V_o t/T_o & \text{for } 0 \leq t \leq T_o, \\ V_o & \text{for } t \geq T_o, \end{cases} \tag{17.20}$$

where T_o indicates how long it takes for the switch to be fully "closed."

The current through the loop is

$$i(t) = C\frac{dv_c(t)}{dt} = \begin{cases} 0 & \text{for } t < 0, \\ CV_o/T_o & \text{for } 0 < t < T_o, \\ 0 & \text{for } t > T_o. \end{cases} \tag{17.21}$$

Notice (17.21) is deliberately vague about what $i(t)$ is at exactly $t = 0$ and $t = T_o$. The energy delivered by the battery is

$$\int_{-\infty}^{\infty} v_c(t)i(t)dt = \int_0^{T_o} \left(\frac{V_o t}{T_o}\right)\left(\frac{CV_o}{T_o}\right)dt = \frac{CV_o^2 t^2}{2T_o^2}\bigg|_{t=0}^{t=T_o} = \frac{CV_o^2}{2}. \tag{17.22}$$

So, under this model, the energy provided by the battery matches the final energy stored in the capacitor, no matter what T_o is.

17.3.3 The True Paradox – and its Solution

The question concerning the missing energy in (17.3), by itself, isn't really the "paradox." The true paradox is that there are so many disparate "solutions" to the paradox, all of which somewhat miss the mark as a true "solution" since they involve the introduction of new factors.

The real underlying issue lies in a bit of sloppiness in the way (17.17) was handled:

$$\int_{-\infty}^{\infty} v_c(t)i(t)dt = \int_{-\infty}^{\infty} [V_0 u(t)][CV_0\delta(t)]dt = \int_{-\infty}^{\infty} V_0^2 C u(t)\delta(t)dt. \tag{17.23}$$

From the earliest pages of this text, we have emphasized that situations in which it matters what u(t) is at $t = 0$ should be treated with great suspicion. Here, it matters, and the last equality of (17.17) of questionable validity since $\int_{-\infty}^{\infty} u(t)\delta(t)dt$ is not a well-formed mathematical expression.

We feel that the most honest explanation of the "capacitor paradox" is to simply note that (17.3) is *not a valid circuit*, in the same way that two voltages sources with different voltages connected in parallel, or two current sources connected in series with different currents, are not valid circuits.