GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

FINAL EXAM

DATE: 27-Apr-18 COURSE: ECE-2026

NAME: Solutions GT ID: ____________

LAST, FIRST (ex: buzz1a)

3 points 3 points 3 points

Recitation Section: Circle the date & time when your **Recitation Section** meets (not Lab):

			L06:Thur-Noon (Fekri)
			L08:Thurs-1:30pm (Fekri)
L01:M-3pm (Valenta)	L09:Tues-3pm (Rohling)	L02:W-3pm (Yang)	L10:Thur-3pm (Marenco)
L03:M-4:30pm (Valenta)	L11:Tues-4:30pm (Rohling)	L04:W-4:30pm (Yang)	L12:Thur-4:30pm (Marenco)

- Write your name on the front page ONLY. **DO NOT** unstaple the test.
- Closed book, but a calculator is permitted.
- One sheet ($8\frac{1}{2}'' \times 11''$) of notes permitted. OK to write on both sides.
- justify your reasoning clearly to receive partial credit.
  Explanations are also required to receive full credit for any answer.
- You must write your answer in the space provided on the exam paper itself.
  Only these answers will be graded. Circle your answers, or write them in the boxes provided.
  If space is needed for scratch work, use the backs of previous pages.

Problem	*Value*	*Score*
1	20	
2	11	
3	10	
4	16	
5	12	
6	14	
7	17	
No/Wrong Rec	-3	

Problem F.1:

Basic Concepts

Each part of this problem is independent of the others.

(a) **[3 pts]** What is the instantaneous frequency of the following signal at time $t = 9$ in Hertz?

$$x(t) = 4\cos(2\pi t + 3\pi\sqrt{t} - \pi/5)$$

$$f(t) = \frac{1}{2\cancel{\pi}}\frac{d}{dt}\left(2\cancel{\pi}t + 3\cancel{\pi}t^{1/2} - \cancel{\pi}/5\right)$$

$$= 1 + \frac{3t^{-1/2}}{4}$$

$$f(9) = 1 + \frac{3}{4}\left(\frac{1}{3}\right) = \boxed{\frac{5}{4}\text{ Hz}}$$

(b) **[3 pts]** Find a real, *positive* amplitude A and a phase ϕ between $-\pi$ and π so that the following equation is true:

$$A\cos(6\pi t - \pi/4) - 0.5\sin(6\pi t + \phi) = 0.5\cos(6\pi t).$$

use phasors

$$Ae^{-j\pi/4} \underbrace{-\frac{1}{2}e^{j(\phi - \pi/2)}}_{+\frac{1}{2}je^{j\phi}} = 1/2$$

real

$$2A\cos(-\pi/4) - \sin\phi = 1$$

$$\sqrt{2}A = 1 - \sin\phi$$

imag

$$2A\sin\left(\frac{-\pi}{4}\right) + \cos\phi = 0$$

$$-\sqrt{2}A = -\cos\phi$$

$$\cos\phi = 1 - \sin\phi$$

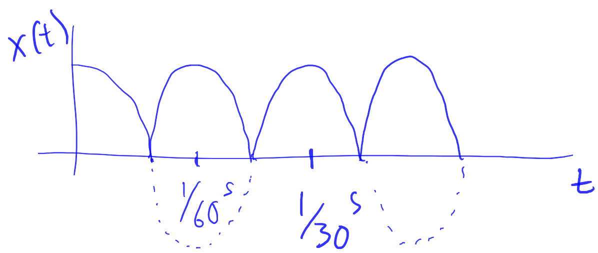
$$\phi = 0$$

$$\boxed{\phi = 0, \quad A = 1/\sqrt{2}}$$

(c) **[2 pts]** What is the length of $y[n] = x[n] * h[n]$ if $x[n]$ and $h[n]$ have lengths 19 and 47 respectively?

$19 + 47 - 1 = \boxed{65}$

(d) **[3 pts]** Suppose $x(t) = |\cos(2\pi \cdot 30t)|$. What is the the fundamental frequency of $x(t)$ in Hertz?

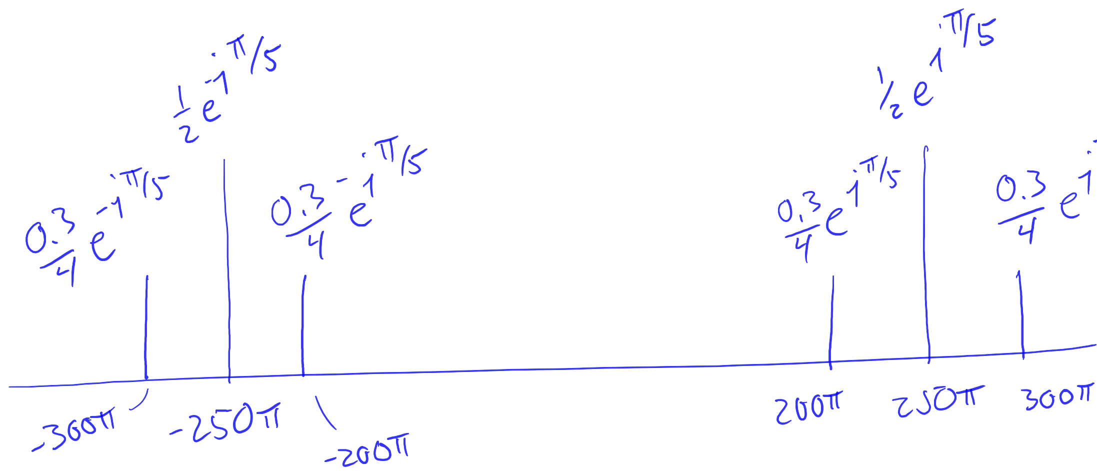


$\boxed{60 \text{ Hz}}$

(e) **[3 pts]** Sketch the spectrum (*not* spectrogram) of the following signal.

$$x(t) = (1 + 0.3\cos(50\pi t))\cos(250\pi t + \pi/5)$$

$x(t) = \cos\left(250\pi t + \pi/5\right) + \frac{0.3}{2}\left(\cos(300\pi t + \pi/5) + \cos\left(200\pi t + \pi/5\right)\right)$



(f) **[3 pts]** Given $H(e^{j\hat{\omega}}) = \frac{1}{1+e^{-2j\hat{\omega}}}$, find a simplified expression for $\left|H(e^{j\hat{\omega}})\right|^2$. (There should be no j's in your answer!)

$$\left|H(e^{j\hat{\omega}})\right|^2 = \frac{1}{1+e^{-2j\hat{\omega}}} \cdot \frac{1}{1+e^{2j\hat{\omega}}} = \frac{1}{1+2\cos(2\hat{\omega})+1}$$

$$= \boxed{\frac{1}{2+2\cos(2\hat{\omega})}}$$

(g) **[3 pts]** Evaluate the following sum and express it in the form $Ae^{j\theta}$.

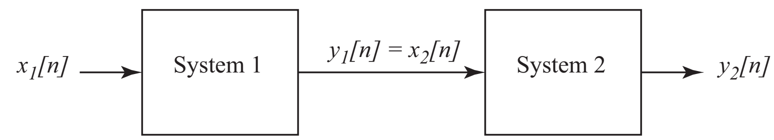
$$z = \sum_{k=0}^{10} e^{j(2\pi k/12+\pi^{-1})}$$

$$z = \underbrace{\cancel{\sum_{k=0}^{11} e^{j(2\pi k/12 + \pi^{-1})}}}_{0} - e^{j(2\pi\,11/12 + \pi^{-1})}$$

$$= \boxed{e^{j\left(2\pi \frac{11}{12} + \pi^{-1} + \pi\right)}}$$

Problem F.2:

Frequency Response



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = -a_2 y_2[n-1] + x_2[n]$$

(a) **[5 pts]** Write an expression for $H(z)$, the system function of the *overall* cascade system.

$$H_1(z) = 1 + b_1 z^{-1} + z^{-2}$$

$$H_2(z) = \frac{1}{1 + a_2 z^{-1}}$$

$$\Rightarrow H_1(z) H_2(z) = \boxed{\frac{1 + b_1 z^{-1} + z^{-2}}{1 + a_2 z^{-1}}}$$

(b) **[6 pts]** When the input to the overall cascade system is the following:

$$x_1[n] = 2(-1)^n + 2\cos(\pi n/2 + \pi/3)$$

the corresponding output of the overall system is:

$$y_2[n] = \cos(\pi n/2 + \phi)$$

Determine the numerical values of b_1.*Show enough work to make it clear how you arrived at your final answer.*

we need a zero at $\hat{\omega} = \pi$ $(z = -1)$

$(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) = 1 + b_1 z^{-1} + z^{-2}$ so $-(\beta_1 + \beta_2) = b_1$, $\beta_1 \beta_2 = 1$

so if $\beta_1 = -1$, then $\beta_2 = -1$ and $\boxed{b_1 = 2}$

Problem F.3:

Frequency Response

[10 pts] (2 pts each) Pick the correct frequency response characteristic and enter the number in the answer box.

(a) $y[n] = \frac{1}{2}y[n-1] + x[n]$

$\boxed{9}$

(b) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

$\boxed{6}$

(c) `yn = filter([1,1],1,xn)`

$\boxed{4}$

(d) $h[n] = \sum_{k=0}^{3} \delta[n-k]$

$\boxed{3}$

(e) $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

$\boxed{5}$

1. $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$

2. $\angle H(e^{j\hat{\omega}}) = -3\hat{\omega}$

3. $H(e^{j\hat{\omega}}) = \dfrac{\sin 2\hat{\omega}}{\sin\frac{1}{2}\hat{\omega}} e^{-j1.5\hat{\omega}}$

4. $\left|H(e^{j\hat{\omega}})\right|^2 = 2 + 2\cos(\hat{\omega})$ *(mag. squared)*

5. $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\left(\frac{1}{2} + \cos\hat{\omega} + \cos 2\hat{\omega}\right)$

6. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 + \frac{1}{2}e^{-j\hat{\omega}}}$

7. $H(e^{j\hat{\omega}}) = 1 - \frac{1}{2}e^{-j\hat{\omega}}$

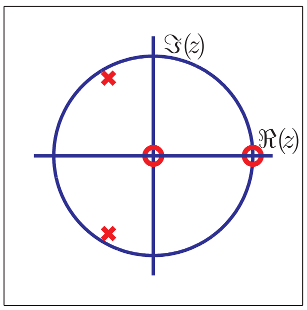
8. $H(e^{j\hat{\omega}}) = \dfrac{\sin\hat{\omega}}{\sin\frac{1}{2}\hat{\omega}}$

9. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}$

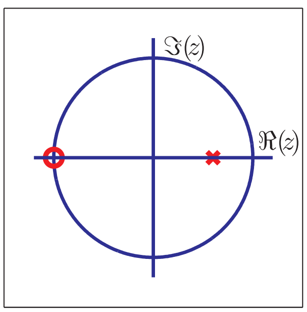
Problem F.4:

Frequency and Impulse Responses

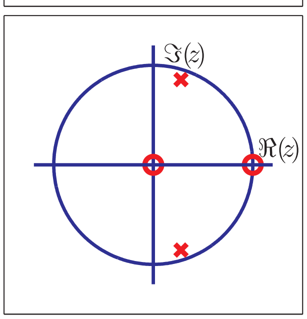
[16 pts] (2 pts each) Below are the pole-zero plots of the z-transforms ($H(z)$) of four discrete-time systems. On the following pages are plots of magnitude frequency responses ($\left|H(e^{j\hat{\omega}})\right|$) and impulse responses ($h[n]$). The numbers on the pole-zero plots represent the multiplicity of the poles and zeros. For each pole-zero plot, enter the letter of the matching frequency response and impulse response respectively. If it is helpful, you can tear the next two pages out of the exam to facilitate comparison, but turn them in with your exam.



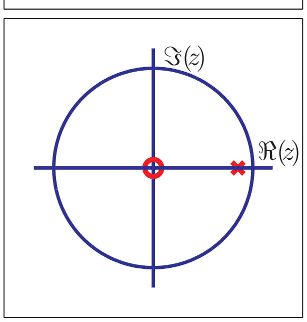
Frequency Response	Impulse Response
ANS = H	**ANS =** l



Frequency Response	Impulse Response
ANS = D	**ANS =** j

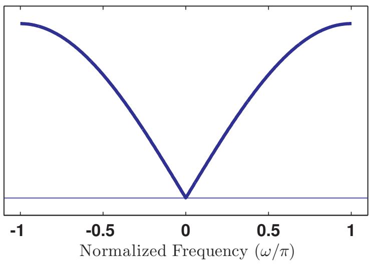


Frequency Response	Impulse Response
ANS = G	**ANS =** n

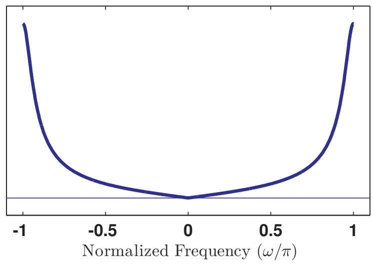


Frequency Response	Impulse Response
ANS = F	**ANS =** k

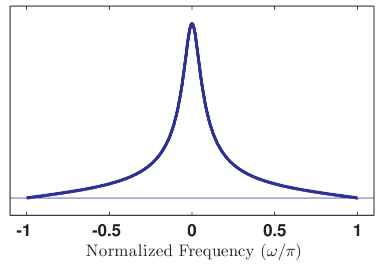
(Magnitude) Frequency Responses



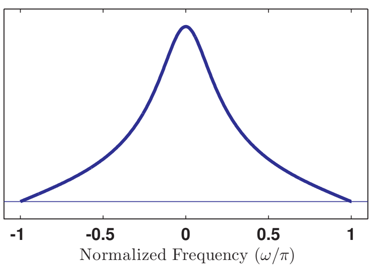
A



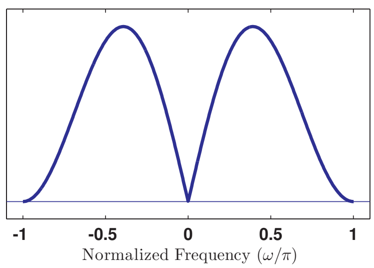
B



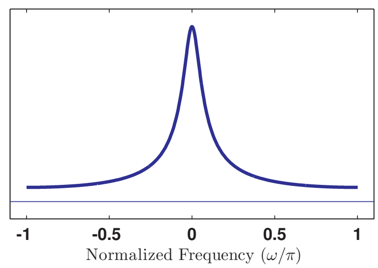
C



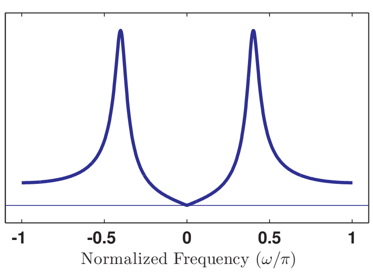
D



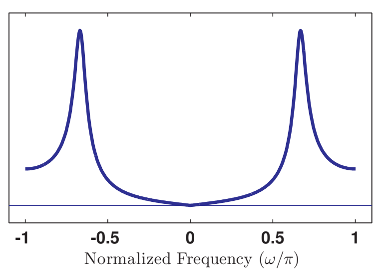
E



F

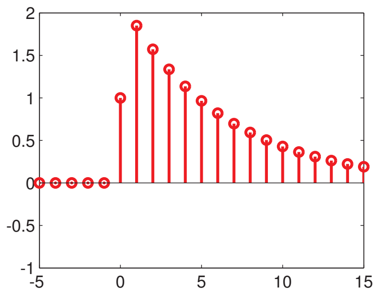


G

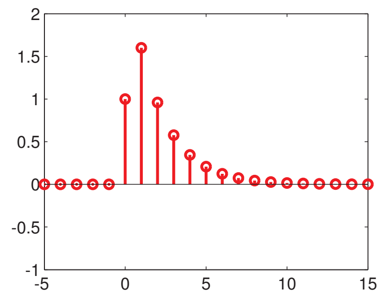


H

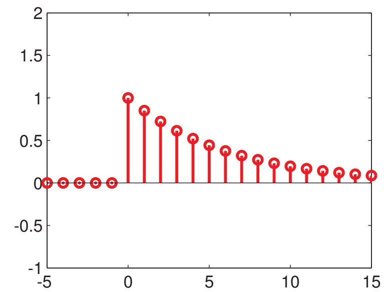
Impulse Responses



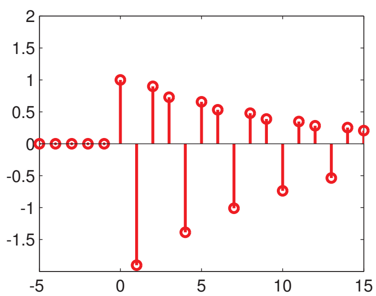
i



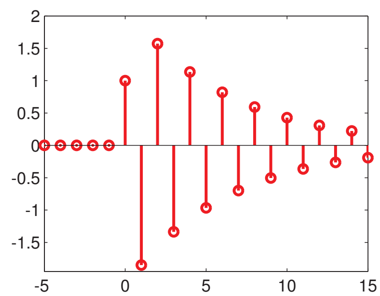
j



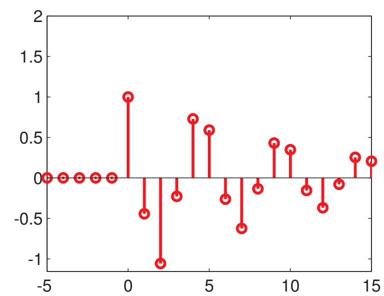
k



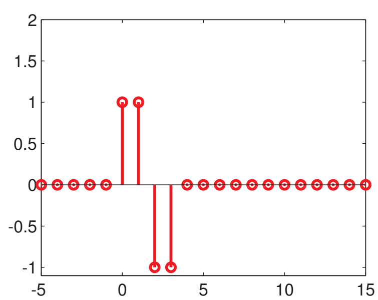
l



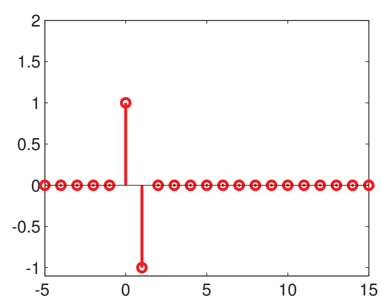
m



n



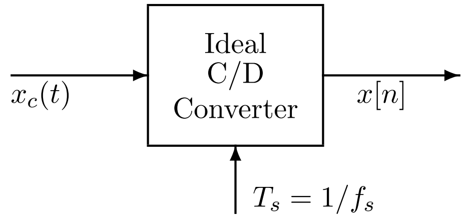
o



p

Problem F.5:

Sampling and Aliasing



For each of the following questions, refer to ideal sampling system above.

(a) **[4 pts]** For a sampling frequency of $f_s = 400$ Hz determine $x[n]$ when

$$x(t) = \cos(300 \cdot 2\pi t + \pi/3).$$

$$x[n] = \cos\left(\frac{300}{400} \cdot 2\pi n + \pi/3\right) = \cos\left(\frac{3}{2}\pi n + \pi/3\right)$$

$$\boxed{x[n] = \cos\left(\frac{\pi}{2} n - \pi/3\right)}$$

(b) **[4 pts]** Determine one value of f_s such that $x[n] = 0$ when

$$x(t) = \cos(400\pi t + \pi/2) + \cos(200\pi t + \pi/2).$$

any $f_s = \frac{200}{k}$ or $\frac{300}{k}$ for $k = 1, 2, \ldots$ works

e.g. 100
150
200
300

(c) **[4 pts]** Suppose, as in part b, $x(t) = \cos(400\pi t + \pi/2) + \cos(200\pi t + \pi/2)$. Determine a general equation for all values of f_s such that $x[n] = 0$. *Hint, at least one of the values is in the range* $100 \le f_s \le 200$.

$$x[n] = \cos\left(\frac{400}{f_s}\pi n + \frac{\pi}{2}\right) + \cos\left(\frac{200}{f_s}\pi n + \pi/2\right)$$

$$= -\sin\left(\frac{400}{f_s}\pi n\right) - \sin\left(\frac{200}{f_s}\pi n\right)$$ $\Rightarrow$ this $= 0$ if $\frac{200}{f_s}$ is a pos. integer

so $\boxed{f_s = \frac{200}{k} \text{ for } k = 1, 2, \ldots}$

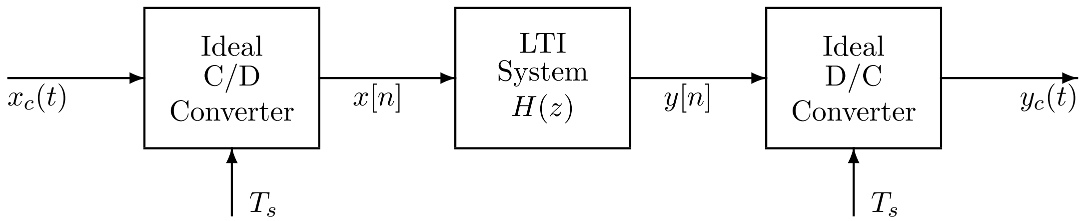
for folding

$$2\pi m + \frac{-400\pi}{f_s} = \frac{200\pi}{f_s} \Rightarrow \frac{600\pi}{f_s} = 2\pi m \Rightarrow \boxed{f_s = \frac{300}{m} \quad m = 1, 2, \ldots}$$

Problem F.6:

Frequency Response

For all parts of this problem, consider the following system for discrete-time filtering of a continuous-time signal.



The system function, $H(z)$, of the discrete-time IIR system can be derived from the difference equation

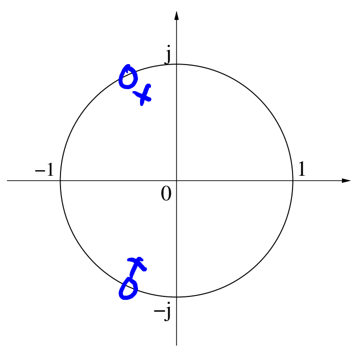
$$y[n] = -0.9\beta y[n-1] - 0.81y[n-2] + x[n] + \beta x[n-1] + x[n-2].$$

(a) **[4 pts]** For $\beta = 1$, determine the poles and zeros of the system and plot your answer on the following pole-zero plot.

$$H(z) = \frac{1 + \beta z^{-1} + z^{-2}}{1 + 0.9\beta z^{-1} + 0.81 z^{-2}}$$

for roots at $\alpha e^{\pm j\theta}$ we have

$1 - 2\alpha \cos\theta\, z^{-1} + \alpha^2 z^{-2}$



for the zeros

$\alpha = 1, \ \theta = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \Rightarrow$

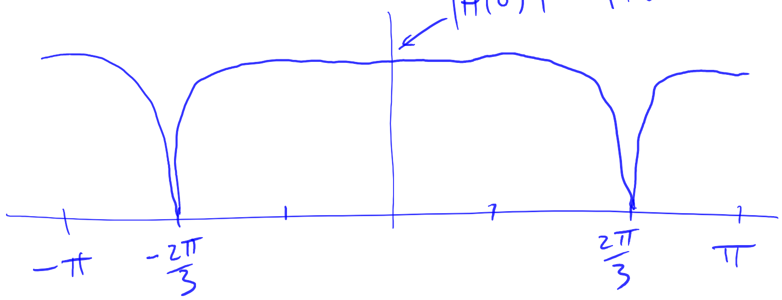
for the poles

$\alpha = 0.9, \ \theta = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \Rightarrow$

$$\boxed{\begin{array}{l} \text{zeros: } e^{\pm j 2\pi/3} \\ \text{poles: } 0.9\, e^{\pm j 2\pi/3} \end{array}}$$

(b) **[5 pts]** This system is a notch filter with the null frequency determined by β. Sketch the magnitude frequency response of this system for $\beta = 1$ over $-\pi \le \hat{\omega} \le \pi$.

$|H(0)| = \frac{1+1+1}{1+0.9+0.81} = 1.11$



(c) **[5 pts]** For a sampling rate of $f_s = 2000$ Hz, determine a new value of β so that the system will null out a frequency component at 500Hz.

$$f_0 = 500 \text{ Hz} \Rightarrow \hat{\omega} = \frac{2\pi 500}{2000} = \frac{\pi}{2}$$

$$\beta = -2\cos\left(\frac{\pi}{2}\right) = 0$$

$$\boxed{\beta = 0}$$

Problem F.7:

DFT Properties

Each part of this problem is independent of the others.

(a) **[4 pts]** Suppose the DFT, $X[k]$, of a sequence $x[n]$ below, is real. That is, $X^*[k] = X[k]$ for $k = 0, \ldots, 6$. Can the unknown values of $x[n]$ be determined? If *yes*, give the missing values. If *no*, then justify your answer.

$$\{1, 7, ?, ?, 6, 5, 7\}$$

since $X^*[k] = X[k]$, $\quad x[N-n] = x^*[n]$

also, for real $x[n]$, $\quad X^*[k] = X[N-k]$

so $x[N-n] = x[n]$ $\quad N = 7$

$$x[n] = \{\underset{n=0}{1}, 7, \underset{n=2}{\boxed{5}}, \underset{n=3}{\boxed{6}}, 6, 5, 7\}$$

(b) **[3 pts]** If $X_8[k]$ is the 8-point DFT of a sequence $\{a, b, c, d, 0, 0, 0, 0\}$, Express $X_4[k]$, the 4-point DFT, of the sequence $\{a, b, c, d\}$ in terms of $X_8[k]$.

$$X_4[k] = X_8[2k]$$

(c) **[3 pts]** Given an unknown length-5 real sequence $x[n]$ with a corresponding 5-point DFT coefficient sequence $X[k] = \{0, 1, j, -j, 1\}$, determine $x[0]$.

$$x[0] = \frac{1}{5} \sum X[k] = \boxed{\frac{2}{5}}$$

(d) **[4 pts]** Given the signal

$$x[n] = \begin{cases} \cos(2\pi n/5) & n = 0, \ldots, 4 \\ 0 & \text{otherwise} \end{cases}$$

Compute the 5-point DFT, $X[k]$, of $x[n]$.

$$X[k] = \sum_{n=0}^{4} \cos(2\pi n/5)\, e^{-j\frac{2\pi kn}{5}}$$

$$= \frac{1}{2}\sum_{n=0}^{4} \left(e^{j2\pi n/5} e^{-j2\pi kn/5} + e^{-j2\pi n/5} e^{-j2\pi kn/5} \right)$$

$$= \frac{1}{2}\sum_{n=0}^{4} \left(e^{j\frac{2\pi n}{5}(1-k)} + e^{-j\frac{2\pi n}{5}(k+1)} \right)$$

the sum is always 0 unless $(1-k) = 5l$ or $(1+k) = 5l$ for integer l

$$\boxed{X[k] = \left\{0, \frac{5}{2}, 0, 0, \frac{5}{2}\right\}}$$

(e) **[3 pts]** For a DFT of length $N = 5$, fill in the table below to show how the indices, k, correspond to the frequencies $\hat{\omega}$ of the DTFT, $X(e^{j\hat{\omega}})$, of $x[n]$.

k	$\hat{\omega}$
0	0
1	$2\pi/5$
2	$4\pi/5$
3	$6\pi/5$ or $\frac{-4\pi}{5}$
4	$\frac{8\pi}{5}$ or $\frac{-2\pi}{5}$

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SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

FINAL EXAM

DATE: April 27, 2018 COURSE: ECE-2026

NAME: ______________________________ GT ID: ______________

LAST, FIRST (ex: buzz2b)

3 points 3 points 3 points

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Problem	*Value*	*Score*
1	20	
2	11	
3	10	
4	16	
5	12	
6	14	
7	17	
No/Wrong Rec	-3	

Problem F.1:

Basic Concepts

Each part of this problem is independent of the others.

(a) **[3 pts]** What is the instantaneous frequency of the following signal at time $t = 9$ in Hertz?

$$x(t) = 4\cos(2\pi t + 3\pi\sqrt{t} - \pi/5)$$

$$f(t) = \frac{1}{2\cancel{\pi}} \frac{d}{dt}\left(2\cancel{\pi} t + 3\cancel{\pi} t^{1/2} - \cancel{\pi}/5\right)$$

$$= 1 + \frac{3t^{-1/2}}{4}$$

$$f(9) = 1 + \frac{3}{4}\left(\frac{1}{3}\right) = \boxed{\frac{5}{4}\ \text{Hz}}$$

(b) **[3 pts]** Find a real, *positive* amplitude A and a phase ϕ between $-\pi$ and π so that the following equation is true:

$$A\cos(6\pi t - \pi/4) - 0.5\sin(6\pi t + \phi) = 0.5\cos(6\pi t).$$

use phasors

$$A e^{-j\pi/4} \underbrace{- \frac{1}{2} e^{j(\phi - \pi/2)}}_{+\frac{1}{2} j e^{j\phi}} = 1/2$$

real

$$2A\cos(-\pi/4) - \sin\phi = 1$$

$$\sqrt{2}A = 1 - \sin\phi$$

imag

$$2A\sin\left(\frac{-\pi}{4}\right) + \cos\phi = 0$$

$$-\sqrt{2}A = -\cos\phi$$

$$\cos\phi = 1 - \sin\phi$$

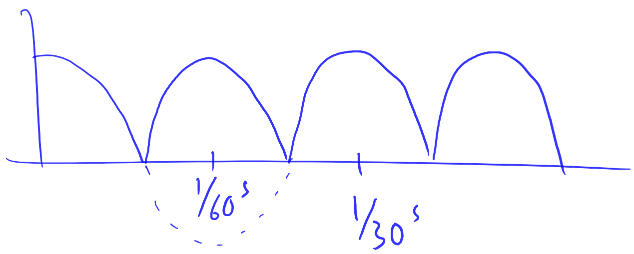
$$\phi = 0$$

$$\boxed{\phi = 0, \quad A = 1/\sqrt{2}}$$

(c) **[2 pts]** What is the length of $y[n] = x[n] * h[n]$ if $x[n]$ and $h[n]$ have lengths 112 and 20 respectively?

$112 + 20 - 1 = \boxed{131}$

(d) **[3 pts]** Suppose $x(t) = |\cos(2\pi \cdot 30t)|$. What is the the fundamental frequency of $x(t)$ in Hertz?



$\boxed{60 \text{ Hz}}$

(e) **[3 pts]** Sketch the spectrum (*not* spectrogram) of the following signal.

$$x(t) = (1 + 0.3\cos(50\pi t))\cos(250\pi t + \pi/5)$$

(f) **[3 pts]** Given $H(e^{j\hat{\omega}}) = \frac{e^{-2j\hat{\omega}}}{1+e^{-2j\hat{\omega}}}$, find a simplified expression for $\left|H(e^{j\hat{\omega}})\right|^2$. (There should be no j's in your answer!)

$$\left|H(e^{j\hat{\omega}})\right|^2 = \frac{e^{-2j\hat{\omega}}}{1+e^{-j2\hat{\omega}}} \cdot \frac{e^{2j\hat{\omega}}}{1+e^{j2\hat{\omega}}} = \frac{1}{1+2\cos(2\hat{\omega})+1}$$

$$= \boxed{\frac{1}{2+2\cos\hat{\omega}}}$$

(g) **[3 pts]** Evaluate the following sum and express it in the form $Ae^{j\theta}$.

$$z = \sum_{k=0}^{8} e^{j(2\pi k/10+\pi^{-1})}$$

$$z = \left(\cancel{\sum_{k=0}^{9} e^{j\left(\frac{2\pi k}{10}+\pi^{-1}\right)}}\right) - e^{j\left(\frac{2\pi \cdot 9}{10}+\pi^{-1}\right)}$$

0

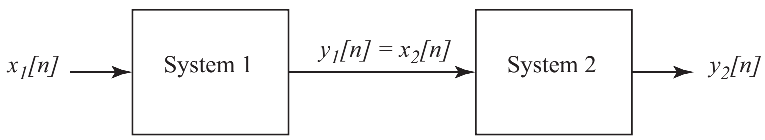
$$= \boxed{e^{j\left(\frac{18\pi}{10}+\pi+\pi^{-1}\right)}}$$

or $e^{j\left(-\frac{\pi}{5}+\pi+\pi^{-1}\right)}$

or $\boxed{e^{j\left(\frac{4\pi}{5}+\pi^{-1}\right)}}$

Problem F.2:

Frequency Response



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = -a_2 y_2[n-1] + x_2[n]$$

(a) **[5 pts]** Write an expression for $H(z)$, the system function of the *overall* cascade system.

$H_1(z) = 1 + b_1 z^{-1} + z^{-2}$

$H_2(z) = \dfrac{1}{1 + a_2 z^{-1}}$ $\Rightarrow H_1(z) H_2(z) = \boxed{\dfrac{1 + b_1 z^{-1} + z^{-2}}{1 + a_2 z^{-1}}}$

(b) **[6 pts]** When the input to the overall cascade system is the following:

$$x_1[n] = 2(-1)^n + 2\cos(\pi n/2 + \pi/3)$$

the corresponding output of the overall system is:

$$y_2[n] = \cos(\pi n/2 + \phi)$$

Determine the numerical values of b_1.*Show enough work to make it clear how you arrived at your final answer.*

we need a zero at $\hat{\omega} = \pi$ $(z = -1)$

$(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) = 1 + b_1 z^{-1} + z^{-2}$ so $-(\beta_1 + \beta_2) = b_1$, $\beta_1 \beta_2 = 1$

so if $\beta_1 = -1$, then $\beta_2 = -1$ and $\boxed{b_1 = 2}$

Problem F.3:

Frequency Response

[10 pts] (2 pts each) Pick the correct frequency response characteristic and enter the number in the answer box.

(a) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

6

(b) $y[n] = \frac{1}{2}y[n-1] + x[n]$

9

(c) `yn = filter([1,1],1,xn)`

4

(d) $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

5

(e) $h[n] = \sum_{k=0}^{3} \delta[n-k]$

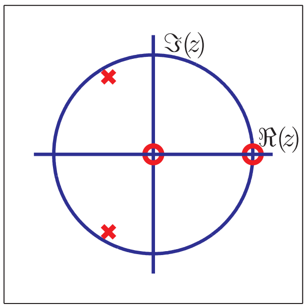
3

1. $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$
2. $\angle H(e^{j\hat{\omega}}) = -3\hat{\omega}$
3. $H(e^{j\hat{\omega}}) = \dfrac{\sin 2\hat{\omega}}{\sin \frac{1}{2}\hat{\omega}} e^{-j1.5\hat{\omega}}$
4. $\left|H(e^{j\hat{\omega}})\right|^2 = 2 + 2\cos(\hat{\omega})$ *(mag. squared)*
5. $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\left(\frac{1}{2} + \cos\hat{\omega} + \cos 2\hat{\omega}\right)$
6. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 + \frac{1}{2}e^{-j\hat{\omega}}}$
7. $H(e^{j\hat{\omega}}) = 1 - \frac{1}{2}e^{-j\hat{\omega}}$
8. $H(e^{j\hat{\omega}}) = \dfrac{\sin\hat{\omega}}{\sin\frac{1}{2}\hat{\omega}}$
9. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}$

Problem F.4:

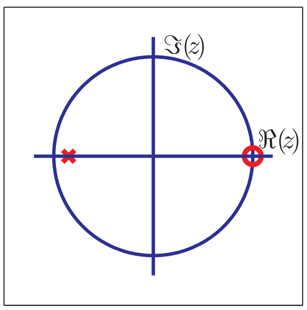
Frequency and Impulse Responses

[16 pts] (2 pts each) Below are the pole-zero plots of the z-transforms ($H(z)$) of four discrete-time systems. On the following pages are plots of magnitude frequency responses ($\left|H(e^{j\hat{\omega}})\right|$) and impulse responses ($h[n]$). The numbers on the pole-zero plots represent the multiplicity of the poles and zeros. For each pole-zero plot, enter the letter of the matching frequency response and impulse response respectively. If it is helpful, you can tear the next two pages out of the exam to facilitate comparison, but turn them in with your exam.



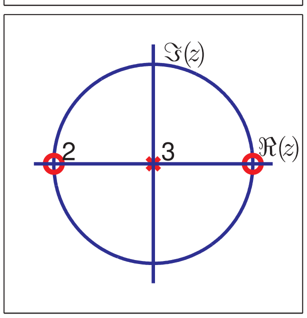
Frequency Response **ANS =** H

Impulse Response **ANS =** l



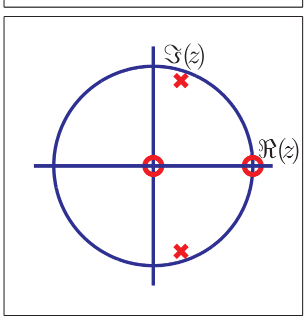
Frequency Response **ANS =** B

Impulse Response **ANS =** m



Frequency Response **ANS =** E

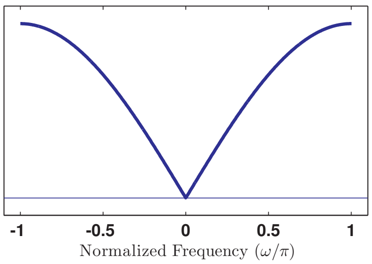
Impulse Response **ANS =** o



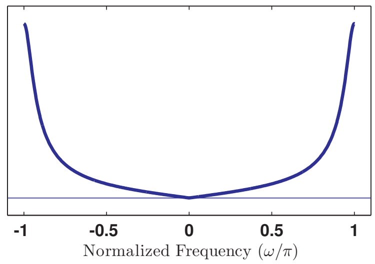
Frequency Response **ANS =** G

Impulse Response **ANS =** n

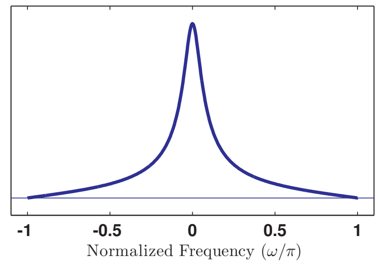
(Magnitude) Frequency Responses



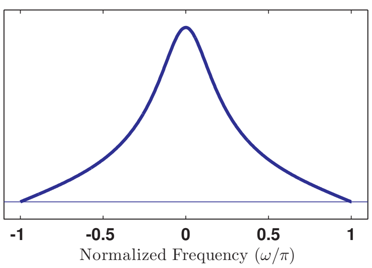
A



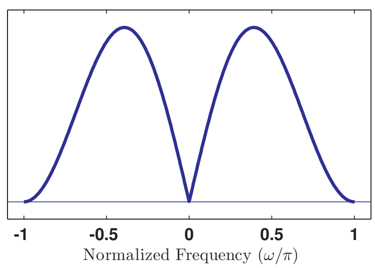
B



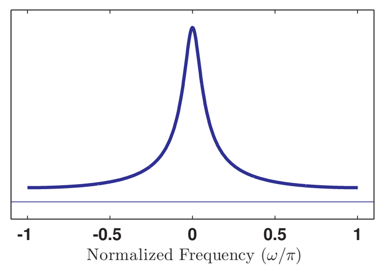
C



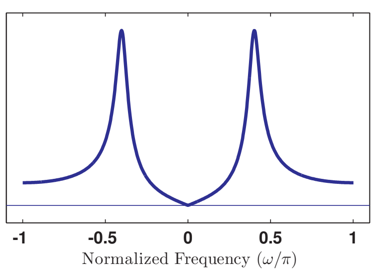
D



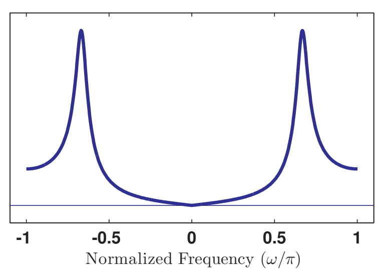
E



F

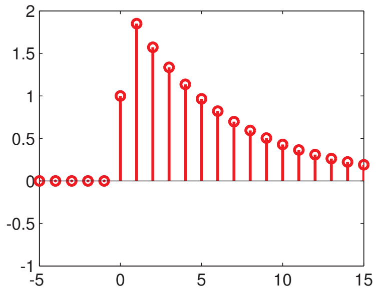


G

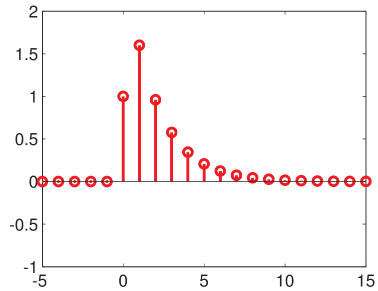


H

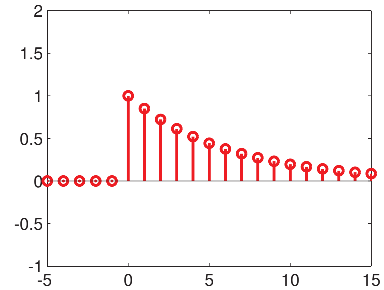
Impulse Responses



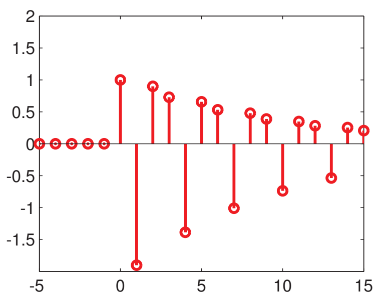
i



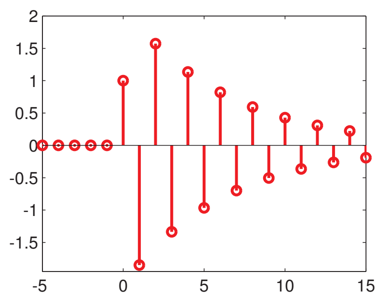
j



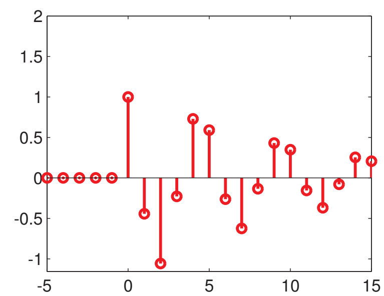
k



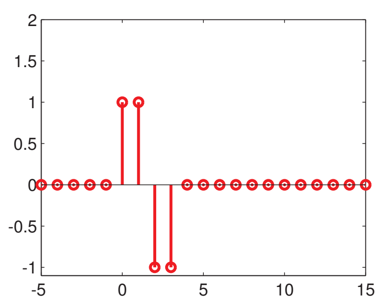
l



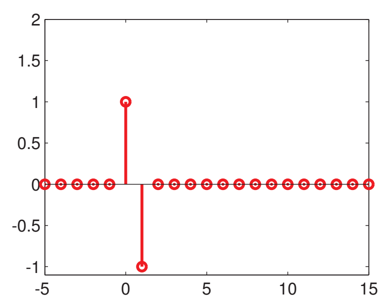
m



n



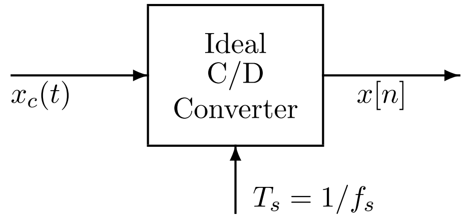
o



p

Problem F.5:

Sampling and Aliasing



For each of the following questions, refer to ideal sampling system above.

(a) **[4 pts]** For a sampling frequency of $f_s = 400$ Hz determine $x[n]$ when

$$x(t) = \cos(250 \cdot 2\pi t + \pi/3).$$

$$x[n] = \cos\left(\frac{250}{400} \cdot 2\pi n + \pi/3\right) = \cos\left(\frac{5\pi}{4} n + \pi/3\right)$$

$$\boxed{x[n] = \cos\left(\frac{3\pi}{4} n - \pi/3\right)}$$

(b) **[4 pts]** Determine one value of f_s such that $x[n] = 0$ when

$$x(t) = \cos(800\pi t + \pi/2) + \cos(1200\pi t + \pi/2).$$

any $f_s = \frac{400}{k}$ or $\frac{1000}{k}$ for $k = 1, 2, \ldots$ works

e.g. 200
250
400
500
1000

(c) **[4 pts]** Suppose, as in part b, $x(t) = \cos(800\pi t + \pi/2) + \cos(1200\pi t + \pi/2)$. Determine a general equation for all values of f_s such that $x[n] = 0$. *Hint, at least one of the values is in the range* $400 \leq f_s \leq 600$.

$$x[n] = -\sin\left(\frac{800}{f_s} \pi n\right) - \sin\left(\frac{1200}{f_s} \pi n\right)$$

we want $\frac{800}{f_s}$ & $\frac{1200}{f_s}$ to be ints.

$$\boxed{f_s = \frac{400}{k}, \quad k = 1, 2, \ldots}$$

$x[n] = 0$ also for $\sin\left(\frac{800}{f_s} \pi n\right) = -\sin\left(\frac{1200 \pi n}{f_s}\right)$

$$= \sin\left(\frac{-1200 \pi n}{f_s} + 2\pi k\right)$$

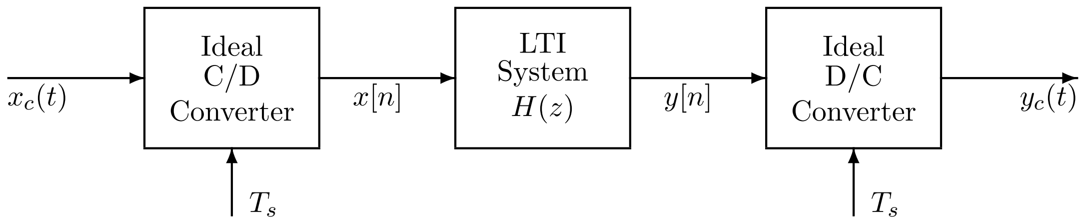
so

$$\frac{800\pi}{f_s} = \frac{-1200\pi}{f_s} + 2\pi k \Rightarrow \boxed{f_s = \frac{1000}{k}, \quad k = 1, 2, \ldots}$$

Problem F.6:

Frequency Response

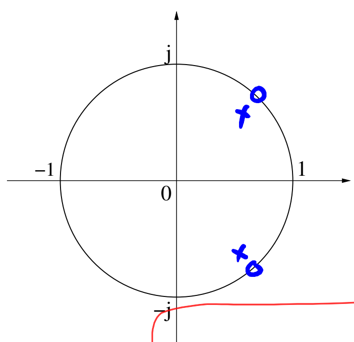
For all parts of this problem, consider the following system for discrete-time filtering of a continuous-time signal.



The system function, $H(z)$, of the discrete-time IIR system can be derived from the difference equation

$$y[n] = -0.9\beta y[n-1] - 0.81 y[n-2] + x[n] + \beta x[n-1] + x[n-2].$$

(a) **[4 pts]** For $\beta = -\sqrt{2}$, determine the poles and zeros of the system and plot your answer on the following pole-zero plot.



roots at $\alpha e^{\pm j\theta} \Rightarrow 1 + 2\alpha\cos\theta\, z^{-1} + \alpha^2 z^{-2}$

$(0.9)\beta = -2\alpha\cos\theta$, $\alpha^2 = 0.81$

$\alpha = 0.9$, $\theta = \cos^{-1}\frac{-\beta}{2} = \cos^{-1}\frac{\sqrt{2}}{2}$

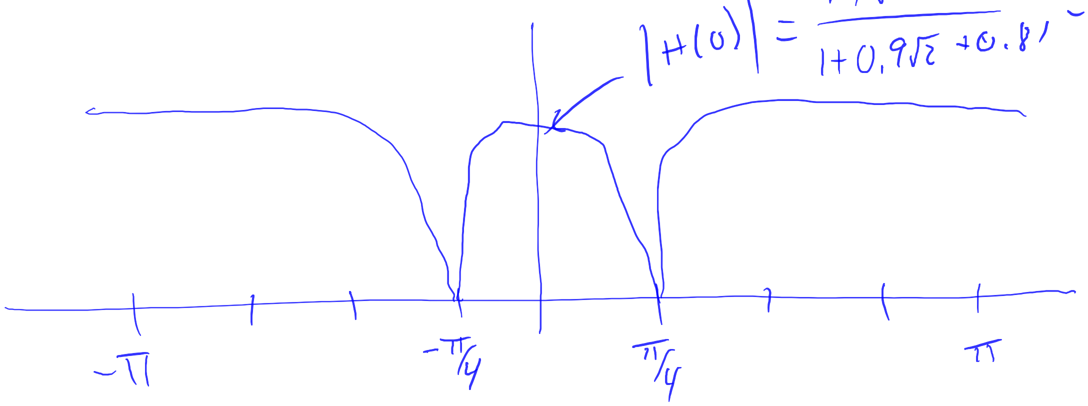
$\theta = \frac{\pi}{4}$

zeros : $e^{\pm j\frac{\pi}{4}}$

poles : $0.9e^{\pm j\pi/4}$

(b) **[5 pts]** This system is a notch filter with the null frequency determined by β. Sketch the magnitude frequency response of this system for $\beta = -\sqrt{2}$ over $-\pi \le \hat{\omega} \le \pi$.

$|H(0)| = \frac{1+\sqrt{2}+1}{1+0.9\sqrt{2}+0.81} = 1.108$



(c) **[5 pts]** For a sampling rate of $f_s = 900$ Hz, determine a new value of β so that the system will null out a frequency component at 300Hz.

$$\hat{\omega}_0 = 2\pi \frac{300}{900} = \frac{2\pi}{3}$$

$$\beta = -2\cos\frac{2\pi}{3} = 1$$

$$\boxed{\beta = 1}$$

Problem F.7:

DFT Properties

Each part of this problem is independent of the others.

(a) **[4 pts]** Suppose the DFT, $X[k]$, of a sequence $x[n]$ below, is real. That is, $X^*[k] = X[k]$ for $k = 0, \ldots, 6$. Can the unknown values of $x[n]$ be determined? If *yes*, give the missing values. If *no*, then justify your answer.

$$\{1, ?, ?, 6, 6, 4, 3\}$$

$$\begin{array}{cccccccc} & n=0 & n=1 & n=2 & & & & \\ x[n] = \{ & 1, & \boxed{3}, & \boxed{4}, & 6, & 6, & 4, & 3 \} \end{array}$$

(b) **[3 pts]** If $X_8[k]$ is the 8-point DFT of a sequence $\{a, b, c, d, 0, 0, 0, 0\}$, Express $X_4[k]$, the 4-point DFT, of the sequence $\{a, b, c, d\}$ in terms of $X_8[k]$.

$$X_4[k] = X_8[2k]$$

(c) **[3 pts]** Given an unknown length-5 real sequence $x[n]$ with a corresponding 5-point DFT coefficient sequence $X[k] = \{0, j, j, -j, -j\}$, determine $x[0]$.

$$x[0] = \frac{1}{5} \sum X[k] = \boxed{0}$$

(d) **[4 pts]** Given the signal

$$x[n] = \begin{cases} \cos(2\pi n/5) & n = 0, \ldots, 4 \\ 0 & \text{otherwise} \end{cases}$$

Compute the 5-point DFT, $X[k]$, of $x[n]$.

$$X[k] = \sum_{n=0}^{4} \cos(2\pi n/5)\, e^{-j\frac{2\pi kn}{5}}$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j2\pi n/5} e^{-j2\pi kn/5} + e^{-j2\pi n/5} e^{-j2\pi kn/5}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j\frac{2\pi n}{5}(1-k)} + e^{-j\frac{2\pi n}{5}(k+1)}\right)$$

the sum is always 0 unless $(1-k)=5l$ or $(1+k)=5l$ for integer l

$$\boxed{X[k] = \left\{0, \frac{5}{2}, 0, 0, \frac{5}{2}\right\}}$$

(e) **[3 pts]** For a DFT of length $N = 5$, fill in the table below to show how the indices, k, correspond to the frequencies $\hat{\omega}$ of the DTFT, $X(e^{j\hat{\omega}})$, of $x[n]$.

k	$\hat{\omega}$
0	0
1	$2\pi/5$
2	$4\pi/5$
3	$6\pi/5$ or $\frac{-4\pi}{5}$
4	$\frac{8\pi}{5}$ or $\frac{-2\pi}{5}$

GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

FINAL EXAM

DATE: 27 APR 18 COURSE: ECE-2026

NAME: ______________________________ GT ID: ________________

LAST, FIRST (ex: buzz3c)

3 points 3 points 3 points

Recitation Section: Circle the date & time when your **Recitation Section** meets (not Lab):

			L06:Thur-Noon (Fekri)
			L08:Thurs-1:30pm (Fekri)
L01:M-3pm (Valenta)	L09:Tues-3pm (Rohling)	L02:W-3pm (Yang)	L10:Thur-3pm (Marenco)
L03:M-4:30pm (Valenta)	L11:Tues-4:30pm (Rohling)	L04:W-4:30pm (Yang)	L12:Thur-4:30pm (Marenco)

- Write your name on the front page ONLY. **DO NOT** unstaple the test.
- Closed book, but a calculator is permitted.
- One sheet ($8\frac{1}{2}'' \times 11''$) of notes permitted. OK to write on both sides.
- justify your reasoning clearly to receive partial credit.
  Explanations are also required to receive full credit for any answer.
- You must write your answer in the space provided on the exam paper itself.
  Only these answers will be graded. Circle your answers, or write them in the boxes provided.
  If space is needed for scratch work, use the backs of previous pages.

Problem	*Value*	*Score*
1	20	
2	11	
3	10	
4	16	
5	12	
6	14	
7	17	
No/Wrong Rec	-3	

Problem F.1:

Basic Concepts

Each part of this problem is independent of the others.

(a) **[3 pts]** What is the instantaneous frequency of the following signal at time $t = 9$ in Hertz?

$$x(t) = 4\cos(2\pi t + 3\pi\sqrt{t} - \pi/5)$$

$$f(t) = \frac{1}{2\cancel{\pi}} \frac{d}{dt}\left(2\cancel{\pi} t + 3\cancel{\pi} t^{1/2} - \cancel{\pi}/5\right)$$

$$= 1 + \frac{3t^{-1/2}}{4}$$

$$f(9) = 1 + \frac{3}{4}\left(\frac{1}{3}\right) = \boxed{\frac{5}{4}\text{ Hz}}$$

(b) **[3 pts]** Find a real, *positive* amplitude A and a phase ϕ between $-\pi$ and π so that the following equation is true:

$$A\cos(6\pi t - \pi/4) - 0.5\sin(6\pi t + \phi) = 0.5\cos(6\pi t).$$

use phasors

$$A e^{-j\pi/4} \underbrace{- \frac{1}{2} e^{j(\phi - \pi/2)}}_{+\frac{1}{2} j e^{j\phi}} = 1/2$$

real

$$2A\cos(-\pi/4) - \sin\phi = 1$$

$$\sqrt{2}A = 1 - \sin\phi$$

imag

$$2A\sin\left(\frac{-\pi}{4}\right) + \cos\phi = 0$$

$$-\sqrt{2}A = -\cos\phi$$

$$\cos\phi = 1 - \sin\phi$$

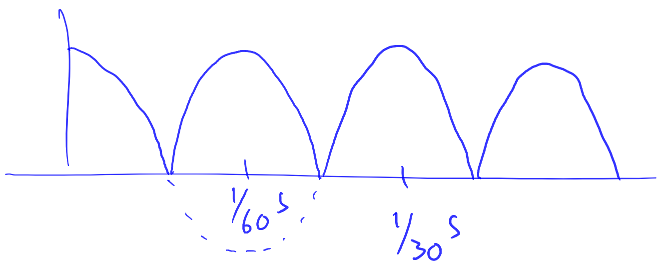
$$\phi = 0$$

$$\boxed{\phi = 0, \quad A = 1/\sqrt{2}}$$

(c) **[2 pts]** What is the length of $y[n] = x[n] * h[n]$ if $x[n]$ and $h[n]$ have lengths 1000 and 100 respectively?

$1000 + 100 - 1 = \boxed{1099}$

(d) **[3 pts]** Suppose $x(t) = |\cos(2\pi \cdot 30t)|$. What is the the fundamental frequency of $x(t)$ in Hertz?



$\boxed{60 \text{ Hz}}$

(e) **[3 pts]** Sketch the spectrum (*not* spectrogram) of the following signal.

$$x(t) = (1 + 0.3\cos(50\pi t))\cos(250\pi t + \pi/5)$$

(f) **[3 pts]** Given $H(e^{j\hat{\omega}}) = \frac{e^{-2j\hat{\omega}}}{1-e^{-3j\hat{\omega}}}$, find a simplified expression for $\left|H(e^{j\hat{\omega}})\right|^2$. (There should be no j's in your answer!)

$$\left|H(e^{j\hat{\omega}})\right|^2 = \frac{e^{-2j\hat{\omega}}}{1-e^{-3j\hat{\omega}}} \cdot \frac{e^{2j\hat{\omega}}}{1-e^{3j\hat{\omega}}}$$

$$= \frac{1}{1-2\cos(3\hat{\omega})+1}$$

$$= \boxed{\frac{1}{2-2\cos(3\hat{\omega})}}$$

(g) **[3 pts]** Evaluate the following sum and express it in the form $Ae^{j\theta}$.

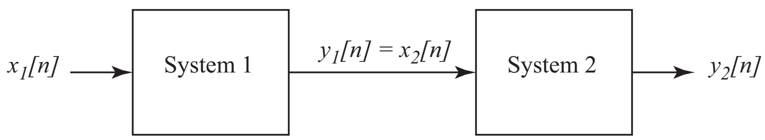
$$z = \sum_{k=0}^{9} e^{j(2\pi k/11+\pi^{-1})}$$

$$z = \underbrace{\left(\sum_{k=0}^{10} e^{j\left(\frac{2\pi k}{11}+\pi^{-1}\right)}\right)}_{0} - e^{j\left(\frac{2\pi\,10}{11}+\pi^{-1}\right)}$$

$$z = \boxed{e^{j\left(\frac{2\pi 10}{11}+\pi+\pi^{-1}\right)}} \quad \text{or} \quad = \boxed{e^{j\left(\frac{9\pi}{11}+\pi^{-1}\right)}}$$

Problem F.2:

Frequency Response



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = -a_2 y_2[n-1] + x_2[n]$$

(a) **[5 pts]** Write an expression for $H(z)$, the system function of the *overall* cascade system.

$$H_1(z) = 1 + b_1 z^{-1} + z^{-2}$$

$$H_2(z) = \frac{1}{1 + a_2 z^{-1}}$$

$$\Rightarrow H_1(z) H_2(z) = \boxed{\frac{1 + b_1 z^{-1} + z^{-2}}{1 + a_2 z^{-1}}}$$

(b) **[6 pts]** When the input to the overall cascade system is the following:

$$x_1[n] = 2(-1)^n + 2\cos(\pi n/2 + \pi/3)$$

the corresponding output of the overall system is:

$$y_2[n] = \cos(\pi n/2 + \phi)$$

Determine the numerical values of b_1.*Show enough work to make it clear how you arrived at your final answer.*

we need a zero at $\hat{\omega} = \pi$ $(z = -1)$

$(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) = 1 + b_1 z^{-1} + z^{-2}$ so $-(\beta_1 + \beta_2) = b_1$, $\beta_1 \beta_2 = 1$

so if $\beta_1 = -1$, then $\beta_2 = -1$ and $\boxed{b_1 = 2}$

Problem F.3:

Frequency Response

[10 pts] (2 pts each) Pick the correct frequency response characteristic and enter the number in the answer box.

(a) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

$\boxed{6}$

(b) $y[n] = \frac{1}{2}y[n-1] + x[n]$

$\boxed{9}$

(c) $h[n] = \sum_{k=0}^{3} \delta[n-k]$

$\boxed{3}$

(d) $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

$\boxed{5}$

(e) `yn = filter([1,1],1,xn)`

$\boxed{4}$

1. $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$

2. $\angle H(e^{j\hat{\omega}}) = -3\hat{\omega}$

3. $H(e^{j\hat{\omega}}) = \dfrac{\sin 2\hat{\omega}}{\sin \frac{1}{2}\hat{\omega}} e^{-j1.5\hat{\omega}}$

4. $\left|H(e^{j\hat{\omega}})\right|^2 = 2 + 2\cos(\hat{\omega})$ *(mag. squared)*

5. $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\left(\frac{1}{2} + \cos\hat{\omega} + \cos 2\hat{\omega}\right)$

6. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 + \frac{1}{2}e^{-j\hat{\omega}}}$

7. $H(e^{j\hat{\omega}}) = 1 - \frac{1}{2}e^{-j\hat{\omega}}$

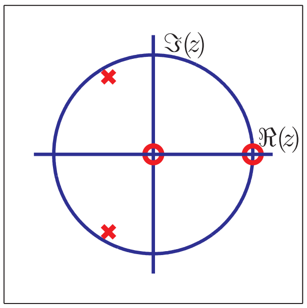
8. $H(e^{j\hat{\omega}}) = \dfrac{\sin\hat{\omega}}{\sin\frac{1}{2}\hat{\omega}}$

9. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}$

Problem F.4:

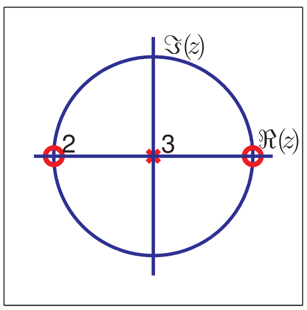
Frequency and Impulse Responses

[16 pts] (2 pts each) Below are the pole-zero plots of the z-transforms ($H(z)$) of four discrete-time systems. On the following pages are plots of magnitude frequency responses ($\left|H(e^{j\hat{\omega}})\right|$) and impulse responses ($h[n]$). The numbers on the pole-zero plots represent the multiplicity of the poles and zeros. For each pole-zero plot, enter the letter of the matching frequency response and impulse response respectively. If it is helpful, you can tear the next two pages out of the exam to facilitate comparison, but turn them in with your exam.



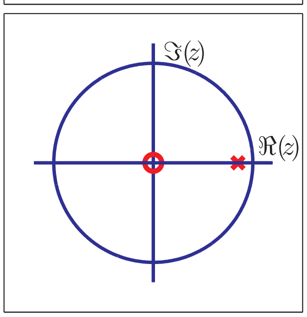
Frequency Response ANS = H

Impulse Response ANS = l



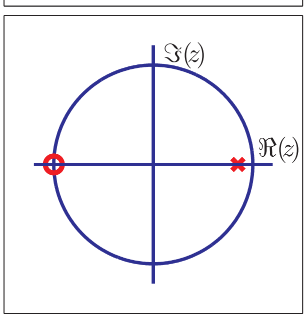
Frequency Response ANS = E

Impulse Response ANS = o



Frequency Response ANS = F

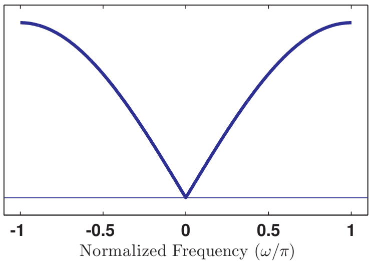
Impulse Response ANS = k



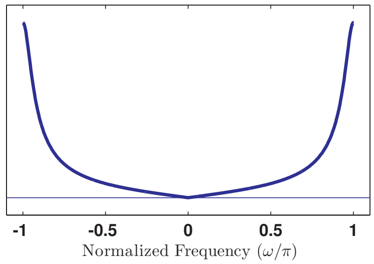
Frequency Response ANS = C

Impulse Response ANS = i

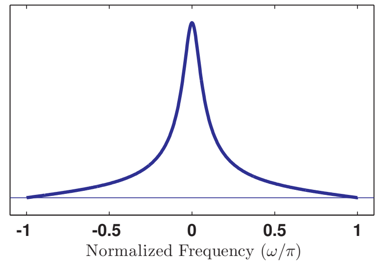
(Magnitude) Frequency Responses



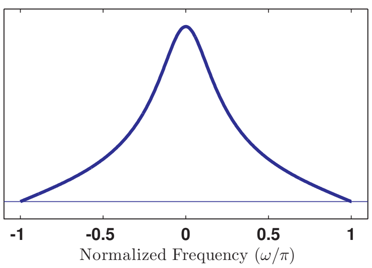
A



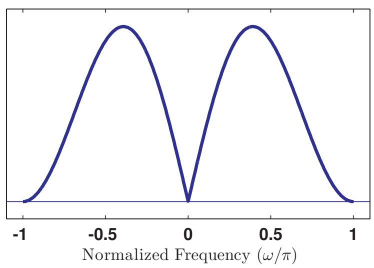
B



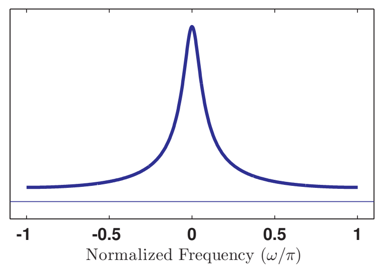
C



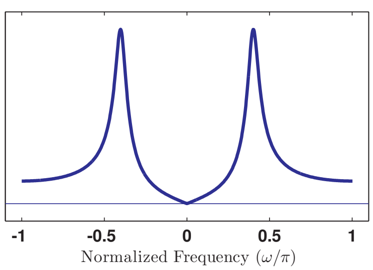
D



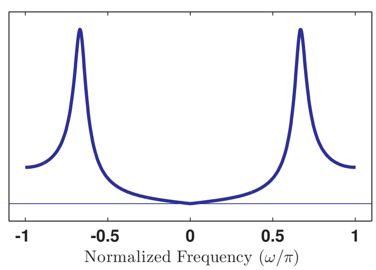
E



F

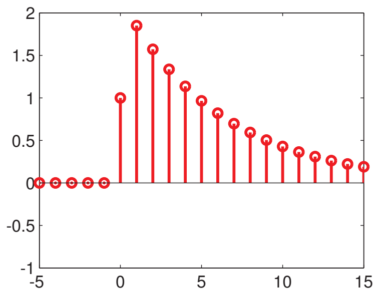


G

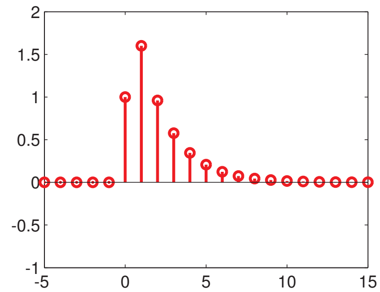


H

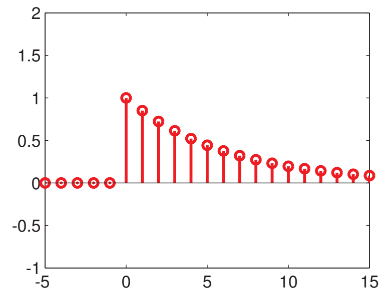
Impulse Responses



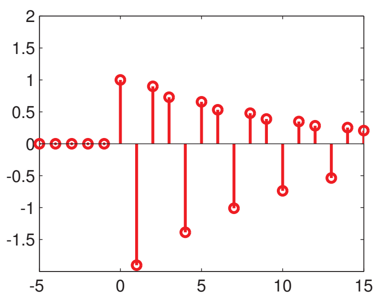
i



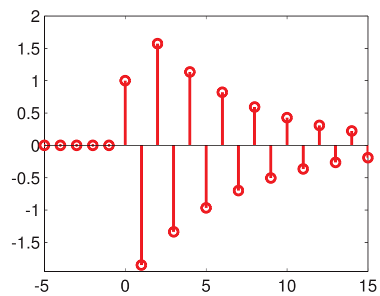
j



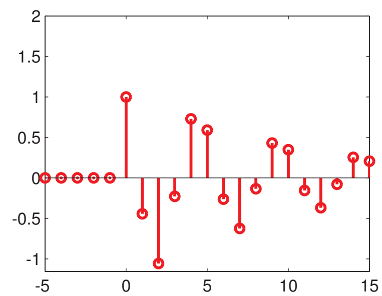
k



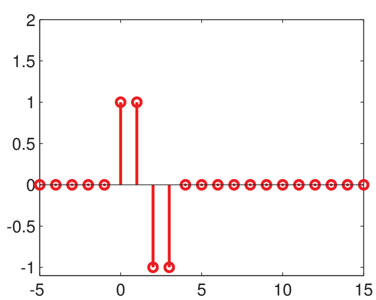
l



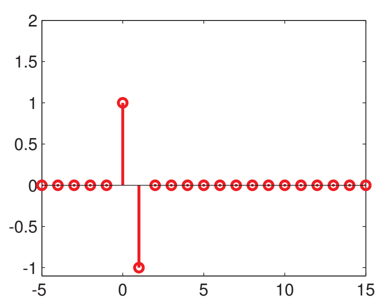
m



n



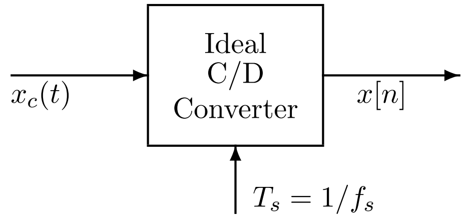
o



p

Problem F.5:

Sampling and Aliasing



For each of the following questions, refer to ideal sampling system above.

(a) **[4 pts]** For a sampling frequency of $f_s = 400$ Hz determine $x[n]$ when

$$x(t) = \cos(250 \cdot 2\pi t + \pi/3).$$

$$x[n] = \cos\left(\frac{250}{400} \cdot 2\pi n + \pi/3\right) = \cos\left(\frac{5}{4}\pi n + \pi/3\right)$$

$$\boxed{x[n] = \cos\left(\frac{3\pi}{4} n - \pi/3\right)}$$

(b) **[4 pts]** Determine one value of f_s such that $x[n] = 0$ when

$$x(t) = \cos(400\pi t + \pi/2) + \cos(800\pi t + \pi/2).$$

$$f_s = \frac{400}{k} \text{ or } \frac{600}{k} \quad \text{for } k = 1, 2, \ldots$$

e.g. 200
300
400
600

(c) **[4 pts]** Suppose, as in part b, $x(t) = \cos(400\pi t + \pi/2) + \cos(800\pi t + \pi/2)$. Determine a general equation for all values of f_s such that $x[n] = 0$. *Hint, at least one of the values is in the range* $200 \leq f_s \leq 400$.

$x[n] = -\sin\left(\frac{400}{f_s}\pi n\right) - \sin\left(\frac{800}{f_s}\pi n\right)$ ⇒ if $\frac{400}{f_s}$ is an integer, both terms $= 0$ ⇒ $\boxed{f_s = \frac{400}{k}, k = 1, 2, \ldots}$

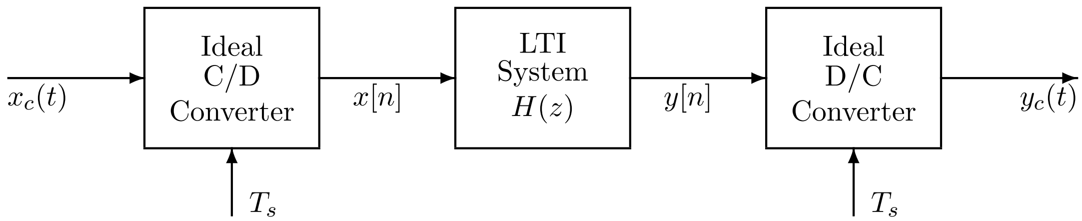
also, if $\sin\left(\frac{400\pi n}{f_s}\right) = -\sin\left(\frac{800}{f_s}\pi n - 2\pi m\right)$ then

$$\frac{400\pi}{f_s} = \frac{-800\pi}{f_s} + 2\pi k \Rightarrow \boxed{f_s = \frac{600}{k}, \; k = 1, 2, \ldots}$$

Problem F.6:

Frequency Response

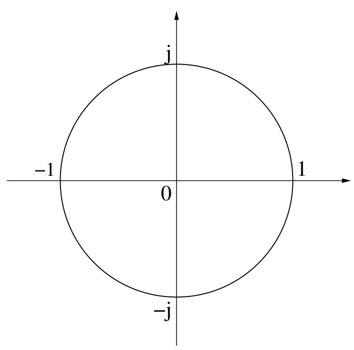
For all parts of this problem, consider the following system for discrete-time filtering of a continuous-time signal.



The system function, $H(z)$, of the discrete-time IIR system can be derived from the difference equation

$$y[n] = -0.9\beta y[n-1] - 0.81y[n-2] + x[n] + \beta x[n-1] + x[n-2].$$

(a) **[4 pts]** For $\beta = \sqrt{2}$, determine the poles and zeros of the system and plot your answer on the following pole-zero plot.



$$H(z) = \frac{1 + \beta z^{-1} + z^{-2}}{1 + 0.9\beta + 0.81 z^{-2}}$$

for roots at $\alpha e^{\pm j\theta}$ we have

$$1 - 2\alpha\cos\theta\, z^{-1} + \alpha^2 z^{-2}$$

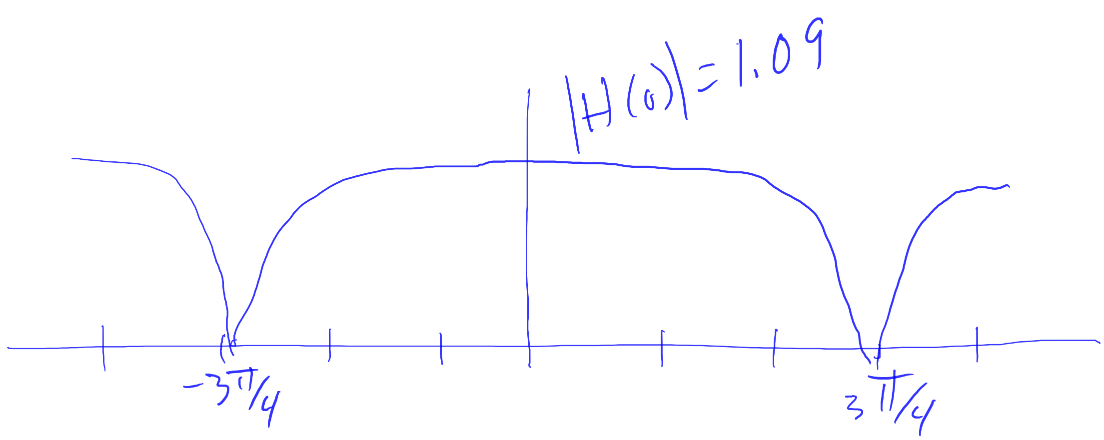
for $\alpha = 0.9$, $-2(0.9)\cos\theta = 0.9\sqrt{2}$

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 3\pi/4$$

zeros : $e^{\pm j3\pi/4}$

poles : $0.9e^{\pm j3\pi/4}$

(b) **[5 pts]** This system is a notch filter with the null frequency determined by β. Sketch the magnitude frequency response of this system for $\beta = \sqrt{2}$ over $-\pi \leq \hat{\omega} \leq \pi$.



(c) **[5 pts]** For a sampling rate of $f_s = 1000$ Hz, determine a new value of β so that the system will null out a frequency component at 125Hz.

$$\hat{\omega}_0 = 2\pi \frac{125}{1000} = \frac{\pi}{4}$$

$$\beta = -2\cos\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

$$\boxed{\beta = -\sqrt{2}}$$

Problem F.7:

DFT Properties

Each part of this problem is independent of the others.

(a) **[4 pts]** Suppose the DFT, $X[k]$, of a sequence $x[n]$ below, is real. That is, $X^*[k] = X[k]$ for $k = 0, \ldots, 6$. Can the unknown values of $x[n]$ be determined? If *yes*, give the missing values. If *no*, then justify your answer.

$$\{1, 7, ?, ?, 7, 1, 7\}$$

$n=0$ $n=2$ $n=3$

$$\{1, 7, \boxed{1}, \boxed{7}, 7, 1, 7\}$$

(b) **[3 pts]** If $X_8[k]$ is the 8-point DFT of a sequence $\{a, b, c, d, 0, 0, 0, 0\}$, Express $X_4[k]$, the 4-point DFT, of the sequence $\{a, b, c, d\}$ in terms of $X_8[k]$.

$$X_4[k] = X_8[2k]$$

(c) **[3 pts]** Given an unknown length-5 real sequence $x[n]$ with a corresponding 5-point DFT coefficient sequence $X[k] = \{-1, -j, 2, 2, j\}$, determine $x[0]$.

$$x[0] = \frac{1}{5} \sum X[k] = \boxed{\frac{3}{5}}$$

(d) **[4 pts]** Given the signal

$$x[n] = \begin{cases} \cos(2\pi n/5) & n = 0, \ldots, 4 \\ 0 & \text{otherwise} \end{cases}$$

Compute the 5-point DFT, $X[k]$, of $x[n]$.

$$X[k] = \sum_{n=0}^{4} \cos(2\pi n/5)\, e^{-j\frac{2\pi kn}{5}}$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j2\pi n/5} e^{-j2\pi kn/5} + e^{-j2\pi n/5} e^{-j2\pi kn/5}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j\frac{2\pi n}{5}(1-k)} + e^{-j\frac{2\pi n}{5}(k+1)}\right)$$

the sum is always 0 unless $(1-k)=5l$ or $(1+k)=5l$ for integer l

$$\boxed{X[k] = \left\{0, \frac{5}{2}, 0, 0, \frac{5}{2}\right\}}$$

(e) **[3 pts]** For a DFT of length $N = 5$, fill in the table below to show how the indices, k, correspond to the frequencies $\hat{\omega}$ of the DTFT, $X(e^{j\hat{\omega}})$, of $x[n]$.

k	$\hat{\omega}$
0	0
1	$2\pi/5$
2	$4\pi/5$
3	$6\pi/5$ or $\frac{-4\pi}{5}$
4	$\frac{8\pi}{5}$ or $\frac{-2\pi}{5}$

GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

FINAL EXAM

DATE: 27-April-18 COURSE: ECE-2026

NAME: ______________________________ GT ID: ______________

LAST, FIRST (ex: buzz4d)

3 points 3 points 3 points

Recitation Section: Circle the date & time when your **Recitation Section** meets (not Lab):

			L06:Thur-Noon (Fekri)
			L08:Thurs-1:30pm (Fekri)
L01:M-3pm (Valenta)	L09:Tues-3pm (Rohling)	L02:W-3pm (Yang)	L10:Thur-3pm (Marenco)
L03:M-4:30pm (Valenta)	L11:Tues-4:30pm (Rohling)	L04:W-4:30pm (Yang)	L12:Thur-4:30pm (Marenco)

- Write your name on the front page ONLY. **DO NOT** unstaple the test.
- Closed book, but a calculator is permitted.
- One sheet ($8\frac{1}{2}'' \times 11''$) of notes permitted. OK to write on both sides.
- justify your reasoning clearly to receive partial credit.
  Explanations are also required to receive full credit for any answer.
- You must write your answer in the space provided on the exam paper itself.
  Only these answers will be graded. Circle your answers, or write them in the boxes provided.
  If space is needed for scratch work, use the backs of previous pages.

Problem	*Value*	*Score*
1	20	
2	11	
3	10	
4	16	
5	12	
6	14	
7	17	
No/Wrong Rec	-3	

Problem F.1:

Basic Concepts

Each part of this problem is independent of the others.

(a) **[3 pts]** What is the instantaneous frequency of the following signal at time $t = 9$ in Hertz?

$$x(t) = 4\cos(2\pi t + 3\pi\sqrt{t} - \pi/5)$$

$$f(t) = \frac{1}{2\cancel{\pi}}\frac{d}{dt}\left(2\cancel{\pi}t + 3\cancel{\pi}t^{1/2} - \cancel{\pi}/5\right)$$

$$= 1 + \frac{3t^{-1/2}}{4}$$

$$f(9) = 1 + \frac{3}{4}\left(\frac{1}{3}\right) = \boxed{\frac{5}{4}\ \text{Hz}}$$

(b) **[3 pts]** Find a real, *positive* amplitude A and a phase ϕ between $-\pi$ and π so that the following equation is true:

$$A\cos(6\pi t - \pi/4) - 0.5\sin(6\pi t + \phi) = 0.5\cos(6\pi t).$$

use phasors

$$Ae^{-j\pi/4} \underbrace{- \frac{1}{2}e^{j(\phi - \pi/2)}}_{+\frac{1}{2}je^{j\phi}} = 1/2$$

real

$$2A\cos(-\pi/4) - \sin\phi = 1$$

$$\sqrt{2}A = 1 - \sin\phi$$

imag

$$2A\sin\left(\frac{-\pi}{4}\right) + \cos\phi = 0$$

$$-\sqrt{2}A = -\cos\phi$$

$$\cos\phi = 1 - \sin\phi$$

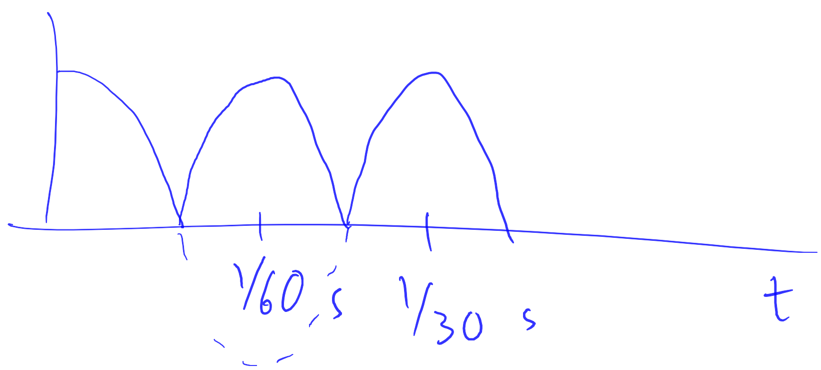
$$\phi = 0$$

$$\boxed{\phi = 0, \quad A = 1/\sqrt{2}}$$

(c) **[2 pts]** What is the length of $y[n] = x[n] * h[n]$ if $x[n]$ and $h[n]$ have lengths 12 and 10 respectively?

$10 + 12 - 1 = \boxed{21}$

(d) **[3 pts]** Suppose $x(t) = |\cos(2\pi \cdot 30t)|$. What is the the fundamental frequency of $x(t)$ in Hertz?



$\boxed{60\ Hz}$

(e) **[3 pts]** Sketch the spectrum (*not* spectrogram) of the following signal.

$$x(t) = (1 + 0.3\cos(50\pi t))\cos(250\pi t + \pi/5)$$

see V.1

(f) **[3 pts]** Given $H(e^{j\hat\omega}) = \frac{2}{1+e^{-3j\hat\omega}}$, find a simplified expression for $\left|H(e^{j\hat\omega})\right|^2$. (There should be no j's in your answer!)

$$\left|H(e^{j\hat\omega})\right|^2 = \frac{2}{1+e^{-j3\hat\omega}} \cdot \frac{2}{1+e^{j3\hat\omega}}$$

$$= \boxed{\frac{4}{2+2\cos(3\hat\omega)}}$$

(g) **[3 pts]** Evaluate the following sum and express it in the form $Ae^{j\theta}$.

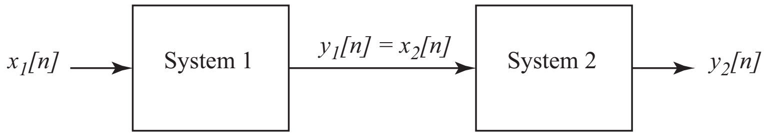
$$z = \sum_{k=0}^{12} e^{j(2\pi k/12+\pi)}$$

$$z = \cancel{\left(\sum_{k=0}^{11} e^{j\left(\frac{2\pi k}{12}+\pi\right)}\right)}^{0} + e^{j\left(\frac{2\pi\, 12}{12}+\pi\right)}$$

$$\boxed{z = e^{j\pi} = -1}$$

Problem F.2:

Frequency Response



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = -a_2 y_2[n-1] + x_2[n]$$

(a) **[5 pts]** Write an expression for $H(z)$, the system function of the *overall* cascade system.

$$H_1(z) = 1 + b_1 z^{-1} + z^{-2}$$

$$H_2(z) = \frac{1}{1 + a_2 z^{-1}}$$

$$\Rightarrow H_1(z) H_2(z) = \boxed{\frac{1 + b_1 z^{-1} + z^{-2}}{1 + a_2 z^{-1}}}$$

(b) **[6 pts]** When the input to the overall cascade system is the following:

$$x_1[n] = 2(-1)^n + 2\cos(\pi n/2 + \pi/3)$$

the corresponding output of the overall system is:

$$y_2[n] = \cos(\pi n/2 + \phi)$$

Determine the numerical values of b_1.*Show enough work to make it clear how you arrived at your final answer.*

we need a zero at $\hat{\omega} = \pi$ $(z = -1)$

$(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) = 1 + b_1 z^{-1} + z^{-2}$ so $-(\beta_1 + \beta_2) = b_1$, $\beta_1 \beta_2 = 1$

so if $\beta_1 = -1$, then $\beta_2 = -1$ and $\boxed{b_1 = 2}$

Problem F.3:

Frequency Response

[10 pts] (2 pts each) Pick the correct frequency response characteristic and enter the number in the answer box.

(a) $h[n] = \sum_{k=0}^{3} \delta[n-k]$

$\boxed{3}$

(b) $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

$\boxed{5}$

(c) $y[n] = \frac{1}{2}y[n-1] + x[n]$

$\boxed{9}$

(d) `yn = filter([1,1,0.5,3,0.5,1,1],1,xn)`

$\boxed{2}$

(e) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

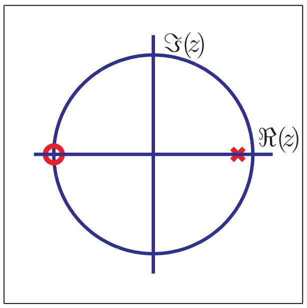
$\boxed{6}$

1. $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$
2. $\angle H(e^{j\hat{\omega}}) = -3\hat{\omega}$
3. $H(e^{j\hat{\omega}}) = \dfrac{\sin 2\hat{\omega}}{\sin \frac{1}{2}\hat{\omega}} e^{-j1.5\hat{\omega}}$
4. $\left|H(e^{j\hat{\omega}})\right|^2 = 2 + 2\cos(\hat{\omega})$ (*mag. squared*)
5. $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\left(\frac{1}{2} + \cos\hat{\omega} + \cos 2\hat{\omega}\right)$
6. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 + \frac{1}{2}e^{-j\hat{\omega}}}$
7. $H(e^{j\hat{\omega}}) = 1 - \frac{1}{2}e^{-j\hat{\omega}}$
8. $H(e^{j\hat{\omega}}) = \dfrac{\sin\hat{\omega}}{\sin\frac{1}{2}\hat{\omega}}$
9. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}$

Problem F.4:

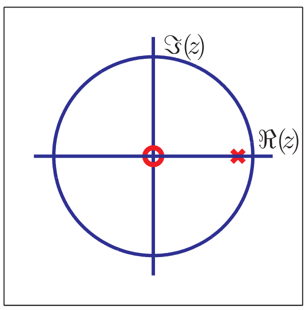
Frequency and Impulse Responses

[16 pts] (2 pts each) Below are the pole-zero plots of the z-transforms ($H(z)$) of four discrete-time systems. On the following pages are plots of magnitude frequency responses ($\left|H(e^{j\hat{\omega}})\right|$) and impulse responses ($h[n]$). The numbers on the pole-zero plots represent the multiplicity of the poles and zeros. For each pole-zero plot, enter the letter of the matching frequency response and impulse response respectively. If it is helpful, you can tear the next two pages out of the exam to facilitate comparison, but turn them in with your exam.



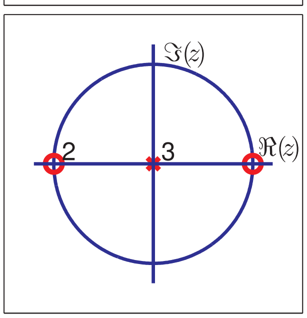
Frequency Response: **ANS =** C

Impulse Response: **ANS =** i



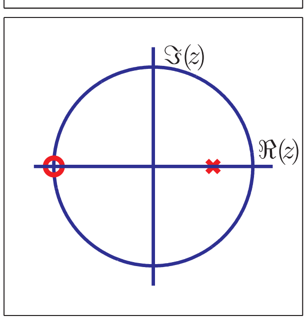
Frequency Response: **ANS =** F

Impulse Response: **ANS =** k



Frequency Response: **ANS =** E

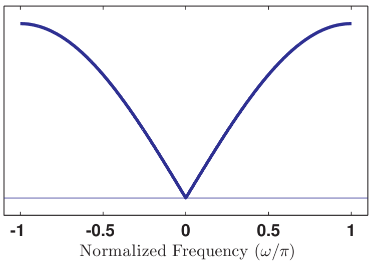
Impulse Response: **ANS =** o



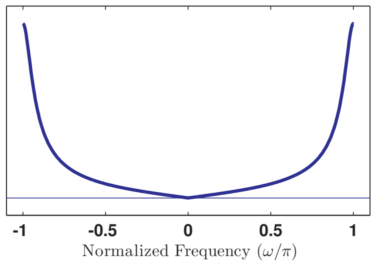
Frequency Response: **ANS =** D

Impulse Response: **ANS =** j

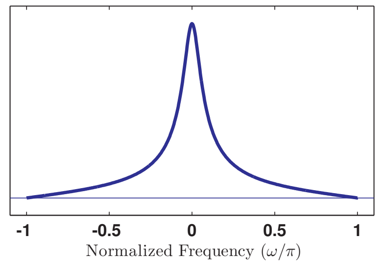
(Magnitude) Frequency Responses



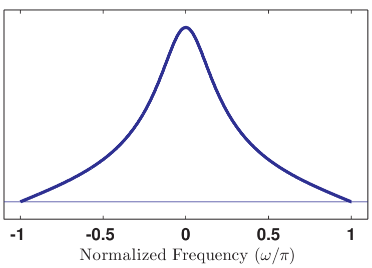
A



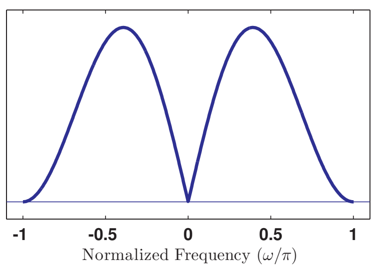
B



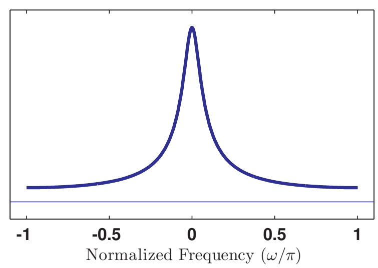
C



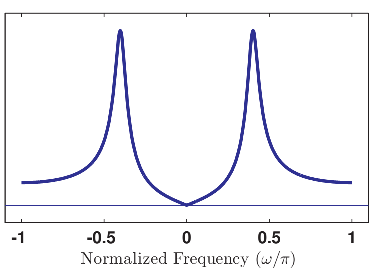
D



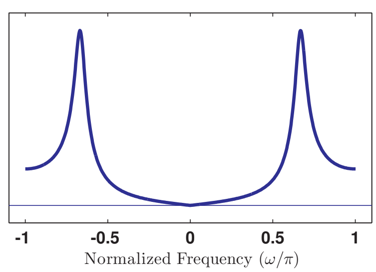
E



F

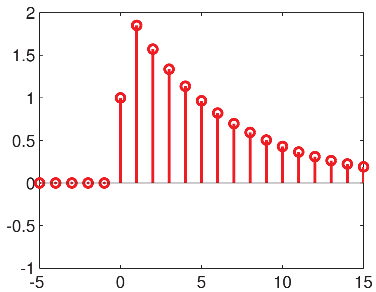


G

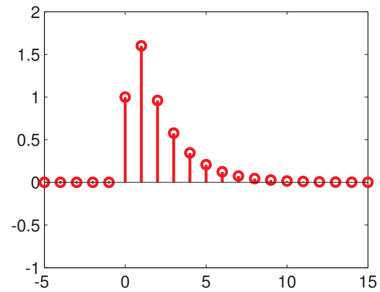


H

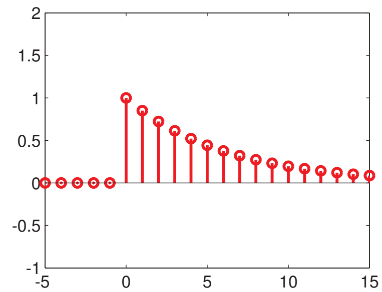
Impulse Responses



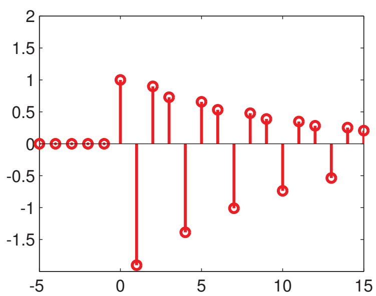
i



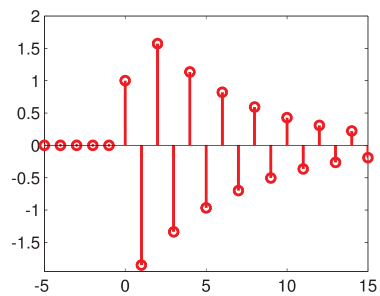
j



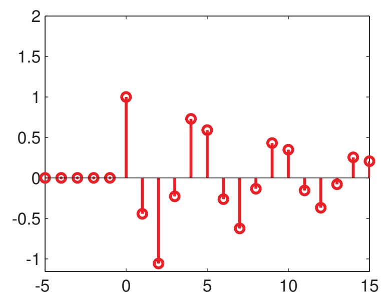
k



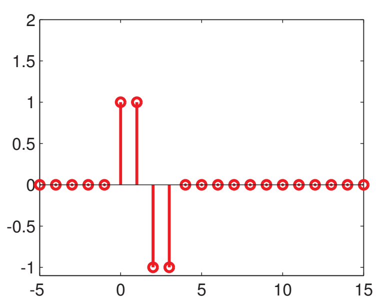
l



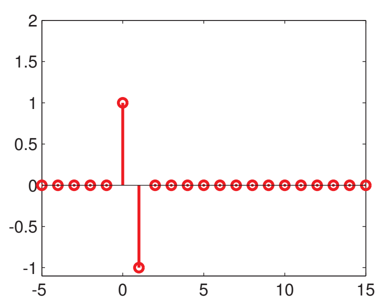
m



n



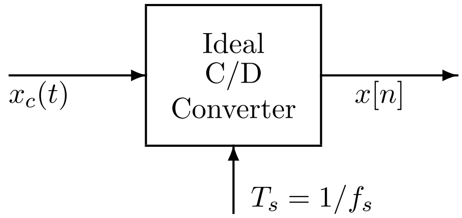
o



p

Problem F.5:

Sampling and Aliasing



For each of the following questions, refer to ideal sampling system above.

(a) **[4 pts]** For a sampling frequency of $f_s = 300$ Hz determine $x[n]$ when

$$x(t) = \cos(350 \cdot 2\pi t + \pi/3).$$

$$x[n] = \cos\left(\frac{350}{300} \cdot 2\pi n + \pi/3\right) = \boxed{\cos\left(\frac{\pi}{3} n + \pi/3\right)}$$

(b) **[4 pts]** Determine one value of f_s such that $x[n] = 0$ when

$$x(t) = \cos(100\pi t + \pi/2) + \cos(50\pi t + \pi/2).$$

$$f_s = \frac{50}{k} \quad \text{or} \quad \frac{75}{k} \quad \text{for} \quad k = 1, 2, \ldots$$

e.g. 25
37.5
50
75

(c) **[4 pts]** Suppose, as in part b, $x(t) = \cos(100\pi t + \pi/2) + \cos(50\pi t + \pi/2)$. Determine a general equation for all values of f_s such that $x[n] = 0$. *Hint, at least one of the values is in the range* $25 \leq f_s \leq 50$.

$$x[n] = \cos\left(\frac{100}{f_s}\pi n + \pi/2\right) + \cos\left(\frac{50\pi}{f_s} n + \pi/2\right) = -\sin\left(\frac{100}{f_s}\pi n\right) - \sin\left(\frac{50}{f_s}\pi n\right)$$

$= 0$ if $\frac{50}{f_s}$ is an integer so $\boxed{f_s = \frac{50}{k}, \ k = 1, 2, \ldots}$

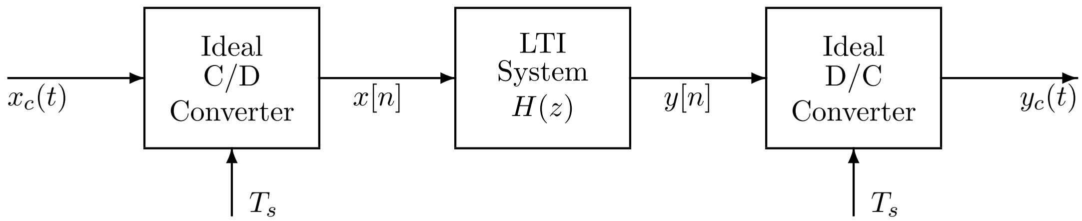
for folding

$$\frac{100}{f_s}\pi = \frac{-50}{f_s}\pi + 2\pi k \Rightarrow \boxed{f_s = \frac{75}{k}, \ k = 1, 2, \ldots}$$

Problem F.6:

Frequency Response

For all parts of this problem, consider the following system for discrete-time filtering of a continuous-time signal.



The system function, $H(z)$, of the discrete-time IIR system can be derived from the difference equation

$$y[n] = -0.9\beta y[n-1] - 0.81y[n-2] + x[n] + \beta x[n-1] + x[n-2].$$

(a) **[4 pts]** For $\beta = -1$, determine the poles and zeros of the system and plot your answer on the following pole-zero plot.

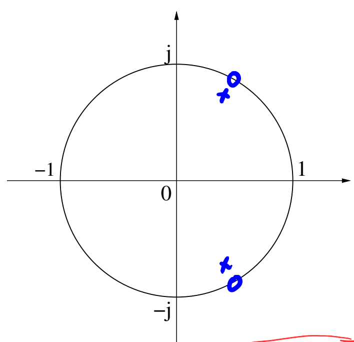
$$H(z) = \frac{1+\beta z^{-1} + z^{-2}}{1+0.9\beta z^{-1} + 0.81 z^{-2}}$$

for roots at $\alpha e^{\pm j\theta}$, we have

$$1 - 2\alpha\cos\theta\, z^{-1} + \alpha^2 z^{-2}$$

$\alpha^2 = 0.81 \Rightarrow \alpha = 0.9$

$0.9\beta = -2(0.9)\cos\theta \Rightarrow \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

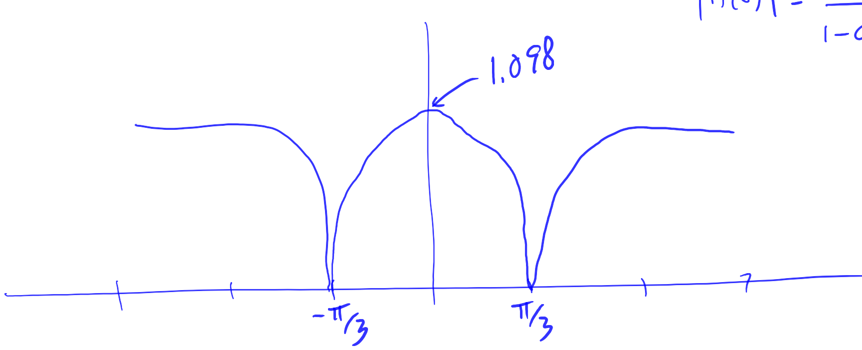


zeros: $e^{\pm j\frac{\pi}{3}}$

poles: $0.9 e^{\pm j\pi/3}$

(b) **[5 pts]** This system is a notch filter with the null frequency determined by β. Sketch the magnitude frequency response of this system for $\beta = -1$ over $-\pi \leq \hat{\omega} \leq \pi$.

$$|H(0)| = \frac{1-1+1}{1-0.9+0.81}$$



(c) **[5 pts]** For a sampling rate of $f_s = 1200$ Hz, determine a new value of β so that the system will null out a frequency component at 200Hz.

$$f_0 = 200\,Hz \qquad \hat{\omega}_0 = \frac{200}{1200} \cdot 2\pi = \frac{\pi}{3}$$

$$\beta = -2\cos\left(\frac{\pi}{3}\right) = -1$$

$$\boxed{\beta = -1}$$

Problem F.7:

DFT Properties

Each part of this problem is independent of the others.

(a) **[4 pts]** Suppose the DFT, $X[k]$, of a sequence $x[n]$ below, is real. That is, $X^*[k] = X[k]$ for $k = 0, \ldots, 6$. Can the unknown values of $x[n]$ be determined? If *yes*, give the missing values. If *no*, then justify your answer.

$$\{1, ?, 0, ?, 6, 0, 7\}$$

$n=0 \quad n=1 \quad n=3$

$$x[n] = \{1, \boxed{7}, 0, \boxed{6}, 6, 0, 7\}$$

(b) **[3 pts]** If $X_8[k]$ is the 8-point DFT of a sequence $\{a, b, c, d, 0, 0, 0, 0\}$, Express $X_4[k]$, the 4-point DFT, of the sequence $\{a, b, c, d\}$ in terms of $X_8[k]$.

$$X_4[k] = X_8[2k]$$

(c) **[3 pts]** Given an unknown length-5 real sequence $x[n]$ with a corresponding 5-point DFT coefficient sequence $X[k] = \{0, 1, -j, j, 1\}$, determine $x[0]$.

$$x[0] = \frac{1}{5} \sum X[k] = \boxed{\frac{2}{5}}$$

(d) **[4 pts]** Given the signal

$$x[n] = \begin{cases} \cos(2\pi n/5) & n = 0, \ldots, 4 \\ 0 & \text{otherwise} \end{cases}$$

Compute the 5-point DFT, $X[k]$, of $x[n]$.

$$X[k] = \sum_{n=0}^{4} \cos(2\pi n/5)\, e^{-j\frac{2\pi kn}{5}}$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j2\pi n/5} e^{-j2\pi kn/5} + e^{-j2\pi n/5} e^{-j2\pi kn/5}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{4}\left(e^{j\frac{2\pi n}{5}(1-k)} + e^{-j\frac{2\pi n}{5}(k+1)}\right)$$

the sum is always 0 unless $(1-k)=5l$ or $(1+k)=5l$ for integer l

$$\boxed{X[k] = \left\{0, \frac{5}{2}, 0, 0, \frac{5}{2}\right\}}$$

(e) **[3 pts]** For a DFT of length $N = 5$, fill in the table below to show how the indices, k, correspond to the frequencies $\hat{\omega}$ of the DTFT, $X(e^{j\hat{\omega}})$, of $x[n]$.

k	$\hat{\omega}$
0	0
1	$2\pi/5$
2	$4\pi/5$
3	$6\pi/5$ or $\frac{-4\pi}{5}$
4	$\frac{8\pi}{5}$ or $\frac{-2\pi}{5}$