GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

FINAL EXAM

DATE: 29-APR-15 COURSE: ECE 2026A

NAME: **Solutions v1** STUDENT #: ____________

LAST, FIRST

2 points 2 points 2 points

Recitation Section: Circle the date & time when your **Recitation Section** meets (not Lab):

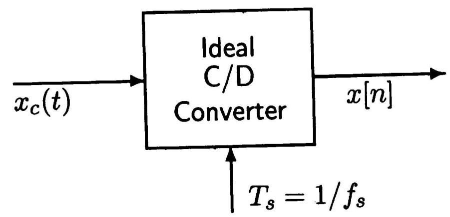
L00:Tue-9:30am (Zhang) L01:Mon-3:00pm (Casinovi)
L03:Mon-4:30pm (Casinovi) L05:Tue-12:00pm (Zhang)
L06:Thu-12:00pm (Walkenhorst) L07:Tue-1:30pm (Zajic)
L08:Thu-1:30pm (Walkenhorst) L09:Tue-3:00pm (Zajic)
L10:Thu-3:00pm (Fekri) L12:Thu-4:30pm (Fekri)

- Write your name on the front page ONLY. **DO NOT** unstaple the test.
- Closed book, but a calculator is permitted. However, one page ($8\frac{1}{2}'' \times 11''$) of HAND-WRITTEN notes permitted. OK to write on both sides.
- Unless stated otherwise, JUSTIFY your reasoning clearly to receive any partial credit. Showing your work is required to receive any partial credit.
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.

Problem	*Value*	*Score*
1	14	
2	14	
3	16	
4	12	
5	14	
6	14	
7	14	
Rec	2	
Total	100	

VI

Problem F.1:
Sampling and Aliasing



For the following questions, refer to ideal sampling system above.

(a) For a sampling frequency of $f_s = 400$ Hz and

$$x(t) = \cos(300 \cdot 2\pi t + \pi/3),$$

determine $x[n]$.

$$x[n] = \cos\left(\frac{300}{400} \cdot 2\pi n + \frac{\pi}{3}\right)$$
$$= \cos\left(1{,}5\pi n + \frac{\pi}{3}\right) = \cos\left(0{,}5\pi n - \frac{\pi}{3}\right)$$

(b) Suppose

$$x(t) = \cos(400\pi t + \pi/2) + \cos(200\pi t + \pi/2)$$

determine one value of f_s such that $x[n] = 0$.

$f_s = 200\,Hz$

$$x[n] = \cos\left(2\pi n + \frac{\pi}{2}\right) + \cos\left(\pi n + \frac{\pi}{2}\right)$$
$$= \emptyset$$

(c) Suppose, as in part b,

$$x(t) = \cos(400\pi t + \pi/2) + \cos(200\pi t + \pi/2)$$

determine a general equation for all values of f_s such that $x[n] = 0$.

$$x[n] = \cos\left(\frac{400}{f_s}\pi n + \frac{\pi}{2}\right) + \cos\left(\frac{200}{f_s}\pi n + \frac{\pi}{2}\right)$$
$$= -\sin\left(\frac{200}{f_s}2\pi n\right) - \sin\left(\frac{100}{f_s}2\pi n\right)$$

$$\frac{200}{f_s} = k \Rightarrow f_s = \frac{200}{k} \quad k = 1, 2, \ldots$$

$$x[n] = \cos\left(\frac{200}{f_s}2\pi n - \frac{\pi}{2}\right) + \cos\left(\frac{100}{f_s}2\pi n + \frac{\pi}{2}\right) = 0$$

$$2\pi\frac{200}{f_s} - 2\pi \ell = -2\pi\frac{100}{f_s} \rightarrow f_s = \frac{300}{\ell} \quad \ell = 1, 2, 3, \ldots$$

Problem F.2:
DFT Properties

(a) Suppose the DFT, $X[k]$, of a sequence $x[n]$ below, is real. That is, $X^*[k] = X[k]$ for $k = 0, \ldots, 7$. Can the unknown values of $x[n]$ be determined? If *yes*, give the missing values. If *no*, then justify your answer.

$$\{1, 7, ?, ?, 1, 6, 5, 7\}$$

(b) Given an unknown length-5 real sequence $x[n]$ with a corresponding 5-point DFT coefficient sequence $X[k] = \{0, 1, j, -j, 1\}$, determine $x[0]$.

(c) If $X_8[k]$ is the 8-point DFT of a sequence $\{a, b, c, d, 0, 0, 0, 0\}$, Express $X_4[k]$, the 4-point DFT, of the sequence $\{a, b, c, d\}$ in terms of $X_8[k]$.

Version 1 for Problem 2

(a) Since X*[k]=X[k], i.e. being real for all *k*, the IDFT of X[k] will also be conjugate symmetric, i.e., x[N-n]=x*[n], but x[n] is real, so we have x[N-n]=x*[n]=x[n]. This can also be shown:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{j2\pi kn/N};$$

$$x[N-n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi k(N-n)/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j2\pi kn/N};$$

$$\text{so } x^*[N-n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{j2\pi kn/N} = x[n] = x[N-n].$$

Therefor if x[n] = {1; 7; ?; ?; 1; 6; 5; 7}, then the missing x[2]=x[8-2]=5, and x[3]=x[8-3]=6.

(b) The IDFT of an N-point DFT sequence, X[k], can be evaluated as:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N};$$

$$\text{so } x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k].$$

Therefor if x[n] = {0; 1; j; -j; 1}, with N=5, then x[0]=2/5.

(c) Recall that zero-padding of a sequence, x[n] = {a; b; c; d}, to y[n] = {a; b; c; d; 0; 0; 0; 0}, produces the same DTFT for both x[n] and y[n]. Therefore, with 8-point DFT that samples the DTFT of y[n], we end up with twice the number of points than that for a 4-point DFT of x[n]. Since there are four sample falling on top of each other, we can deduct that X[k] = Y[2k], for k=0, 1, 2, and 3. This can also be proved as follows:

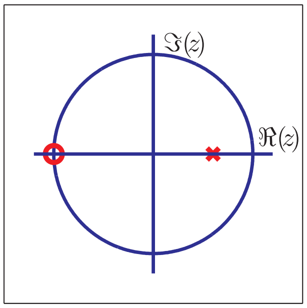
$$Y[k] = X_8[k] = \sum_{n=0}^{7} y[n]e^{j2\pi kn/8};$$

$$\text{and } X[k] = X_4[k] = \sum_{n=0}^{3} x[n]e^{j2\pi kn/4} = \sum_{n=0}^{7} y[n]e^{j2\pi kn/4};$$

$$\text{therefore } X_4[k] = X_8[2k].$$

Problem F.3:
Frequency Response

Below are the pole-zero plots of the z-transforms ($H(z)$) of four discrete-time systems. On the following pages are plots of magnitude frequency responses ($\left|H(e^{j\hat{\omega}})\right|$) and impulse responses ($h[n]$). The numbers on the pole-zero plots represent the multiplicity of the poles and zeros. For each pole-zero plot, enter the letter of the matching frequency response and impulse response respectively. If it is helpful, you can tear the next two pages out of the exam to facilitate comparison, but turn them in with your exam.

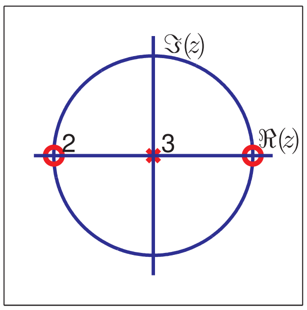


Frequency Response

ANS = A

Impulse Response

ANS = j

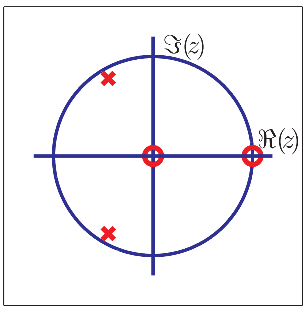


Frequency Response

ANS = F

Impulse Response

ANS = k

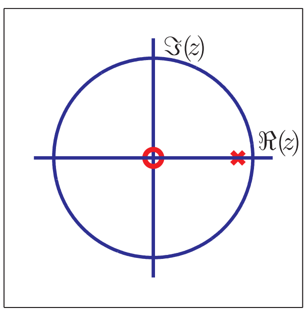


Frequency Response

ANS = B

Impulse Response

ANS = l



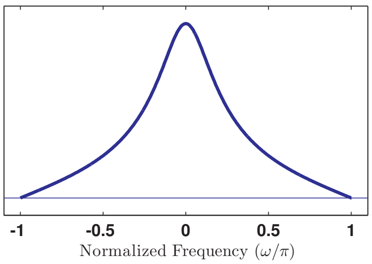
Frequency Response

ANS = C

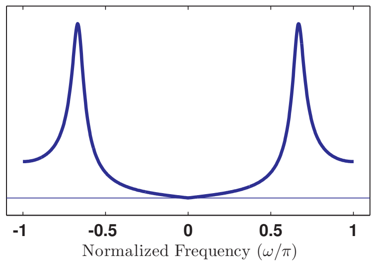
Impulse Response

ANS = i

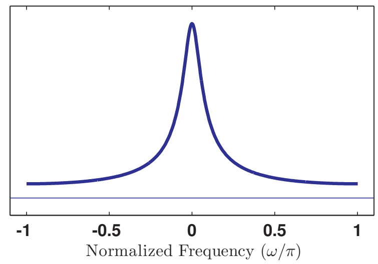
(Magnitude) Frequency Responses



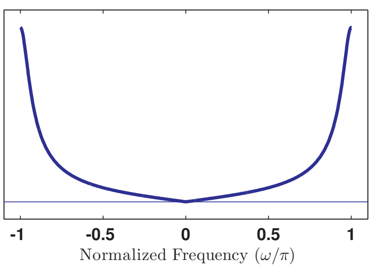
A



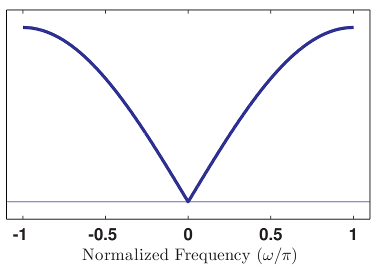
B



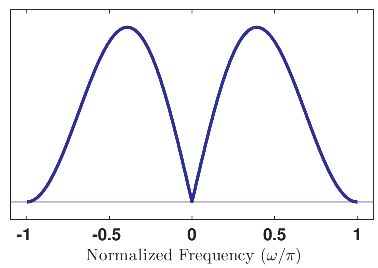
C



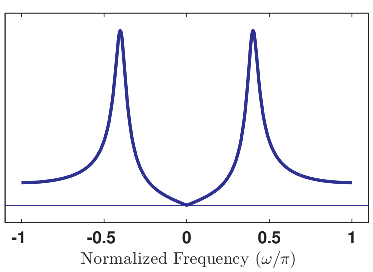
D



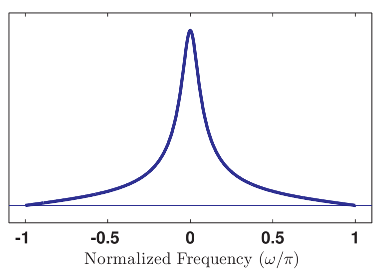
E



F

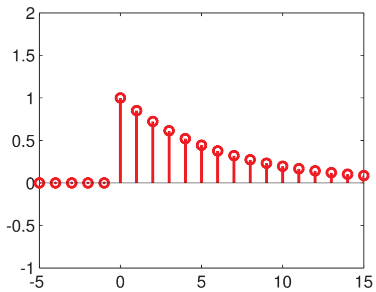


G

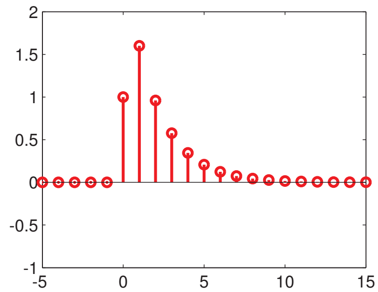


H

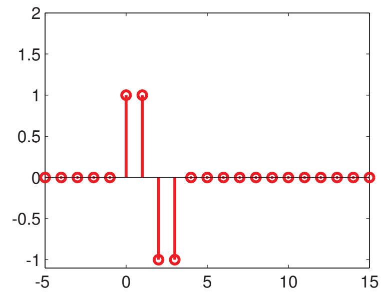
Impulse Responses



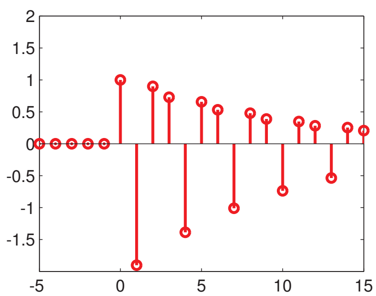
i



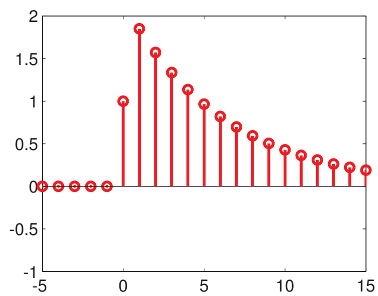
j



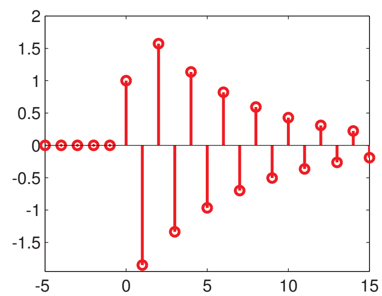
k



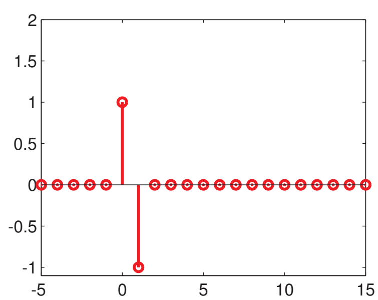
l



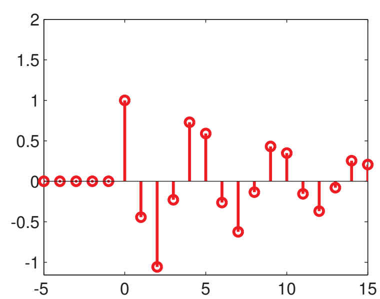
m



n



o



p

Problem F.4:
Frequency Response

Pick the correct frequency response characteristic and enter the number in the answer box:

(a) $h[n] = \sum_{k=0}^{3} \delta[n-k]$

$\cancel{y[n] = \frac{1}{2}y[n-1] + x[n]}$

7

(b) $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

9

(c) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

2

(d) $H(z) = 1 + z^{-1} + z^{-2}$

3

(e) `yn = filter([1,1],1,xn)`

1

(f) $y[n] = \frac{1}{2}y[n-1] + x[n]$

4

1. $\left|H(e^{j\hat{\omega}})\right|^2 = 2 + 2\cos(\hat{\omega})$ (*mag. squared*)

2. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 + \frac{1}{2}e^{-j\hat{\omega}}}$

3. $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$

4. $H(e^{j\hat{\omega}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}$

5. $H(e^{j\hat{\omega}}) = 1 - \frac{1}{2}e^{-j\hat{\omega}}$

6. $\angle H(e^{j\hat{\omega}}) = -2\hat{\omega}$

7. $H(e^{j\hat{\omega}}) = \dfrac{\sin 2\hat{\omega}}{\sin \frac{1}{2}\hat{\omega}} e^{-j1.5\hat{\omega}}$

8. $H(e^{j\hat{\omega}}) = \dfrac{\sin \hat{\omega}}{\sin \frac{1}{2}\hat{\omega}}$

9. $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\left(\frac{1}{2} + \cos\hat{\omega} + \cos 2\hat{\omega}\right)$

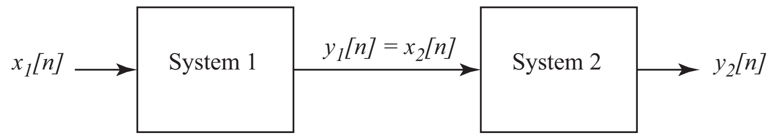
$H(z) = 1 + z^{-1}$

$H(e^{j\omega}) = 1 + e^{-j\omega}$

$= e^{-j\omega/2}\, 2\cos\frac{\omega}{2}$

to get 5, suggestion: change [1 1] to [1 $-\frac{1}{2}$]

Problem F.5:
Frequency Response



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = a_2 y_2[n-1] + x_2[n]$$

(a) Write an expression for $H(z)$, the system function of the *overall* cascade system.

(b) When the input to the system is the following:

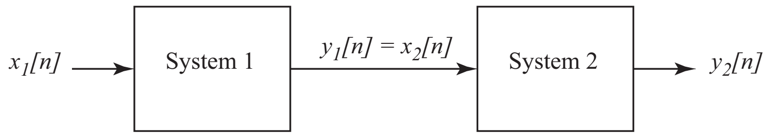
$$x_1[n] = 2(-1)^n - \cos(\pi n/5 + \pi/4)$$

the corresponding output of the overall system is:

$$y_2[n] = 4(-1)^n$$

Determine the numerical values of b_1 and a_2. *Show enough work to make it clear how you arrived at your final answer.* (Hint: Start by computing b_1 based on the fact that one of the input components is missing from the output.)

Problem X



Two systems are connected in cascade, as shown in the figure above. The first system is described by the following difference equation:

$$y_1[n] = x_1[n] + b_1 x_1[n-1] + x_1[n-2]$$

The second system is described by the following difference equation:

$$y_2[n] = a_2 y_2[n-1] + x_2[n]$$

(a) Write an expression for $H(z)$, the system function of the *overall* cascade system.

(b) When the input to the system is the following:

$$x_1[n] = 2(-1)^n - \cos(\pi n/5 + \pi/4)$$

the corresponding output of the overall system is:

$$y_2[n] = 3.5(-1)^n$$

Determine the numerical values of b_1 and a_2. *Show enough work to make it clear how you arrived at your final answer.* (Hint: Start by computing b_1 based on the fact that one of the input components is missing from the output.)

Solution

(a)

$$H(z) = \frac{1 + b_1 z^{-1} + z^{-2}}{1 - a_2 z^{-1}}$$

(b)

$$\begin{aligned}
0 &= 1 + b_1 e^{-j\pi/5} + e^{-j2\pi/5} \\
b_1 &= -\frac{1 + e^{-j2\pi/5}}{e^{-j\pi/5}} = -\left(e^{j\pi/5} + e^{-j\pi/5}\right) \\
&= -2\cos(\pi/5) = -1.618 \\
H(-1) &= \frac{1 - b_1 + 1}{1 + a_2} = \frac{3.5}{2} = 1.75 \\
1 + a_2 &= (2 - b_1)/1.75 = 2.067 \\
a_2 &= 1.067
\end{aligned}$$

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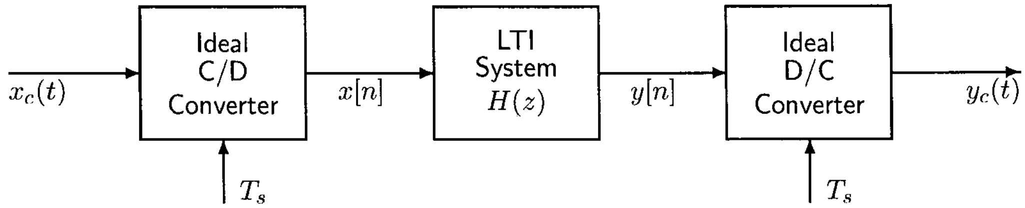
Problem F.6:
Frequency Response

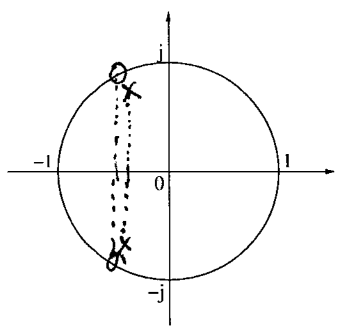
Consider the following system for discrete-time filtering of a continuous-time signal.



The system function, $H(z)$, of the discrete-time IIR system can be derived from the difference equation

$$y[n] = -0.8\alpha y[n-1] - 0.64 y[n-2] + x[n] + \alpha x[n-1] + x[n-2].$$

(a) For $\alpha = 1$, determine the poles and zeros of the system and plot your answer on the following pole-zero plot.



$$H(z) = \frac{1+\alpha z^{-1} + z^{-2}}{1+0.8\alpha z^{-1} + 0.64 z^{-2}} = \frac{(z-z_q)(z-z_q^*)}{(z-z_p)(z-z_p^*)}$$

($z_q + z_q^*$ are zeros)
($z_p + z_p^*$ are poles)

$$= \frac{z^2 - 2Re\{z_q\}z + |z_q|^2}{z^2 - 2Re\{z_p\}z + |z_p|^2} = \frac{z^2 + \alpha z + 1}{z^2 + 0.8\alpha z + 0.64}$$

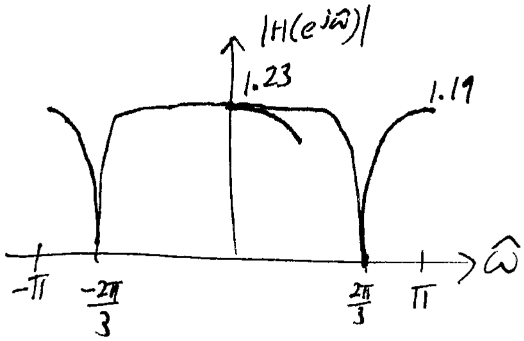
$|z_q| = 1 \quad Re\{z_q\} = -\frac{\alpha}{2} = \frac{-1}{2} \quad \angle z_q = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$|z_p| = 0.8 \quad Re\{z_p\} = -0.4\alpha = -0.4 \quad \angle z_p = \cos^{-1}\left(\frac{-0.4}{-0.8}\right) = \frac{2\pi}{3}$

zero locations: $\boxed{e^{j2\pi/3},\ e^{-j2\pi/3}}$

pole locations: $\boxed{0.8e^{j2\pi/3},\ 0.8e^{-j2\pi/3}}$

(b) This system is a notch filter with the null frequency determined by α. Sketch the magnitude frequency response of this system for $\alpha = 1$ over $-\pi \le \hat{\omega} \le \pi$.



ver 1.

(c) For a sampling rate of $f_s = 2000$ Hz, determine a new value of α so that the system will null out a frequency component at 500Hz.

$$\hat{\omega}_{null} = 2\pi \frac{500}{2000} = \frac{\pi}{2} \qquad z_q = e^{j\frac{\pi}{2}} \qquad z_p = 0.8e^{j\frac{\pi}{2}}$$

$$Re\{z_q\} = -\frac{\alpha}{2} = 0 \Rightarrow \boxed{\alpha = 0}$$

Problem F.7:
Basic Concepts

(a) Find a real, positive amplitude A and a phase ϕ between $-\pi$ and π so that the following equation is true:

$$A\cos(6\pi t - \pi/4) - \sin(6\pi t + \phi) = \cos(6\pi t).$$

$$Ae^{-j\frac{\pi}{4}} - e^{j(\phi - \frac{\pi}{2})} = 1$$

$$\Rightarrow \begin{array}{l} \frac{\sqrt{3}}{2}A - \sin\phi = 1 \\ -\frac{\sqrt{2}}{2}A + \cos\phi = 0 \end{array} \Rightarrow \begin{array}{l} A = \sqrt{2} \\ \phi = 0 \end{array}$$

(b) What is the instantaneous frequency of the following signal at time $t = 4$?

$$x(t) = 4\cos(2\pi t + 3\pi\sqrt{t} + \pi/5)$$

$$\omega_i(t) = \frac{d}{dt}(2\pi t + 3\pi\sqrt{t} + \pi/5) = 2\pi + \frac{3\pi}{2}t^{-\frac{1}{2}}$$

At $t = 4$,

$$\omega_i(t=4) = 2\pi + \frac{3}{4}\pi = 2.75\pi \quad \text{rad/sec}$$

$$f_i(t=4) = \frac{1}{2\pi}\omega_i(t=4) = 1.375 \quad \text{Hz}$$

(c) Suppose $x(t) = |\cos(2\pi \cdot 300t)|$. What is the the fundamental frequency of $x(t)$ in Hertz? (Be sure to notice those absolute value bars!)

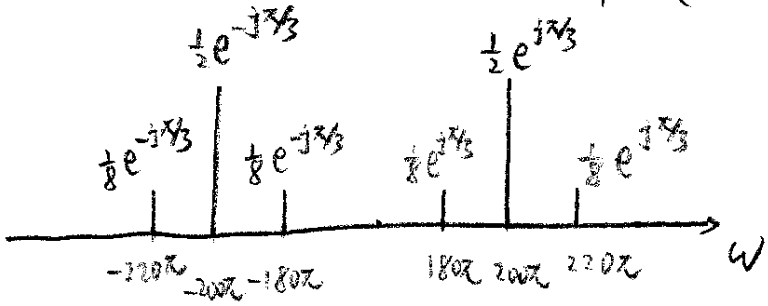
600 Hz

V1

(d) Sketch the spectrum (*not* spectrogram) of the following signal.

$$x(t) = (1 + 0.5\cos(20\pi t))\cos(200\pi t + \pi/3)$$

$$= \cos(200\pi t + \pi/3) + \frac{1}{4}\cos(180\pi t + \pi/3) + \frac{1}{4}\cos(220\pi t + \pi/3)$$



(e) What is the length of $y[n] = x[n] * h[n]$ if $x[n]$ and $h[n]$ have lengths 19 and 47 respectively?

$19 + 47 - 1 = 65$

(f) Given $H(e^{j\hat{\omega}}) = \frac{1}{1+e^{-2j\hat{\omega}}}$, find a simplified expression for $|H(e^{j\hat{\omega}})|$. (There should be no j's in your answer since it is a magnitude!)

$$|H(e^{j\hat{\omega}})| = \sqrt{\frac{1}{(1+\cos 2\hat{\omega})^2 + (\sin 2\hat{\omega})^2}}$$

$$= \sqrt{\frac{1}{2 + 2\cos(2\hat{\omega})}}$$

$$= \sqrt{\frac{1}{2 + 2(2\cos^2\hat{\omega} - 1)}}$$

$$= \frac{1}{2\cos\hat{\omega}}$$