The Impact of Carrier Frequency Offset on Second-Order Algorithms for Blind Channel Identification and Equalization

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Abstract — Recent research has shown that second-order statistics (SOS) are sufficient to blindly identify or equalize a broad class of channels. We consider the effects of carrier frequency offset and determine the criteria under which SOS are still sufficient for channel identification, equalization, and carrier recovery. We show that while equalization may still be possible, channel identification and carrier recovery require use of higher-order statistics. We show that SOS-based channel identification is possible only with knowledge of the carrier frequency offset, and conversely, that SOS-based carrier-offset identification is possible only with knowledge of the channel. We describe algorithms and present simulation results to demonstrate these claims.

I. Introduction

It is well known that higher-order statistics (HOS) are necessary to blindly identify a channel when its input is stationary [1-4]. However, when the channel input is cyclostationary, as is common in digital communications, recent research has shown that second-order statistics (SOS) are sufficient to blindly identify a broad class of channels [5-7]. SOS-based algorithms exploit cyclostationarity by making multiple observations per baud [8]. Because SOS generally require less data than HOS to estimate accurately, SOS-based algorithms can have fast convergence. For example, Tong, Xu, and Kailath [6] have proposed an algorithm (the TXK algorithm) requiring on the order of 100 symbols for accurate channel estimation. Furthermore, SOS-based algorithms permit the use of Gaussian or near-Gaussian input statistics, a necessity at transmission rates approaching capacity.

Classical blind equalization techniques in the presence of frequency offset exploit the HOS of the channel output. For example, a popular technique is first to eliminate the intersymbol interference using a blind equalizer adapted according to the constant modulus algorithm (CMA) [9], and then to track the frequency offset using a carrier recovery loop [10].

In this paper we introduce a carrier frequency offset into the channel model and consider whether SOS alone are sufficient for channel identification, equalization, and carrier recovery in the presence of this offset. In the baud-spaced, stationary case, we can adopt a frequency-domain perspective. Frequency offset shifts the channel output spectrum. It is a simple result that, provided the channel input is white, shifting the channel output spectrum and shifting the spectrum of the channel itself will produce identical spectra at the receiver. Thus, observation of this received spectrum alone will not permit a receiver to distinguish between the two scenarios. This intuition can be extended to the fractionally-spaced, cyclostationary case as well.

We show that *identification* of the channel or the frequency offset requires knowledge of HOS, but that channel *equalization* (removal of intersymbol interference) is still possible. We show that, in a second-order statistical framework, the tasks of channel identification and carrier recovery are coupled. SOS-based channel identification is possible if and only if the carrier frequency offset is known; and conversely, SOS-based carrier recovery is possible only with knowledge of the channel.

We demonstrate the first claim by analyzing the well-known TXK algorithm in the presence of carrier drift and show that it converges not to an estimate of the channel, but rather to an estimate of what we term the *impairment function*. We propose modifications to TXK that make implicit use of HOS to determine the carrier frequency offset, and then use this information to correct the channel estimate.

We also present novel and purely SOS-based algorithms for estimating offset when the channel is known. While using knowledge of the channel, these techniques do not exploit the discrete nature of the constellation, and thus are applicable to dense constellations or even a Gaussian distribution. This technique is applicable to baud-spaced systems as well.

This paper is organized as follows. In section II we describe the channel model with carrier frequency offset. In section III we present theoretical results establishing the criteria under which second-order statistics are sufficient for channel identification, equalization, and carrier recovery. In section IV we analyze the behavior of the TXK algorithm in the presence of carrier offset, and propose a modification to the TXK algorithm to account for the offset. In section V, we propose two second-order offset estimation algorithms for cases in which the channel is known. In section VI we present simulation results.

II. Channel Model

Assume the output of a noisy channel with carrier frequency offset is oversampled L times per baud, and the equivalent baseband discrete-time channel is FIR with impulse response h_i having n taps, as shown in Fig. 1. To model carrier offset, the FIR filter output is multiplied by $e^{j\theta i}$, where $\theta = 2\pi\Delta f T/L$ is the phase drift per sample, Δf is the carrier offset between the modu-

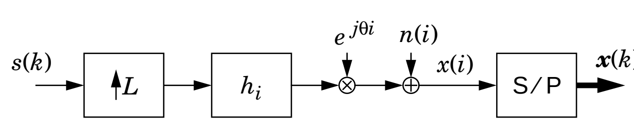

Fig. 1. Channel model with carrier offset.

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lating and demodulating oscillators, and T is the baud interval. Thus, the carrier drift per baud is $\omega = L\theta$. If θ happens to be a rational multiple of π, the channel output $x(i)$ will be strictly cyclostationary. In any case, if the channel input has zero mean, the output will be wide-sense cyclostationary. The symbol sequence $s(k)$ and additive noise $n(i)$ are assumed to be stationary, white, and statistically independent.

As shown in Fig. 1, assume the receiver operates on a batch of m samples and groups them into a vector:

$$\boldsymbol{x}(k) = [x(kL), x(kL-1), \ldots x(kL-m+1)]^T. \qquad (1)$$

Following the development in [11], $\boldsymbol{x}(k)$ can be expressed as:

$$\boldsymbol{x}(k) = e^{j\theta k}\Theta\boldsymbol{H}\boldsymbol{s}(k) + \boldsymbol{n}(k), \qquad (2)$$

where

$$\Theta = \mathrm{diag}[1, e^{-j\theta}, e^{-j2\theta}, \ldots e^{-j(m-1)\theta}], \qquad (3)$$

$$\boldsymbol{H} = \begin{bmatrix} h_0 & h_L & h_{2L} & \cdots & h_{(d-1)L} \\ 0 & h_{L-1} & h_{2L-1} & \cdots & h_{(d-1)L-1} \\ 0 & h_{L-2} & h_{2L-2} & \cdots & h_{(d-1)L-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & h_{L-m+1} & h_{2L-m+1} & \cdots & h_{(d-1)L-m+1} \end{bmatrix}, \qquad (4)$$

$$\boldsymbol{s}(k) = [s(k), s(k-1), \ldots s(k-d+1)]^T, \qquad (5)$$

$$\boldsymbol{n}(k) = [n(kL), n(kL-1), \ldots n(kL-m+1)]^T. \qquad (6)$$

The channel matrix $\boldsymbol{H}$ is of dimension $m \times d$, where d is the number of scalar input samples contributing to $\boldsymbol{x}(k)$. It can also be viewed as the dimension of the signal space defined by the span of $\boldsymbol{H}$. The advantage of this model is that $\boldsymbol{x}(k)$ is wide-sense stationary when the channel input is zero mean. For the model to be valid, we require that $m \geq L \geq 2$. Given m and the number of channel taps n, the length of the vector $\boldsymbol{s}(k)$ is given by [11]:

$$d = \left\lfloor \frac{n+m-2}{L} + 1 \right\rfloor. \qquad (7)$$

We will say more about the proper choice of m in the discussion of the algorithms in section IV.

III. Sufficiency of Second-Order Statistics

We will show that second-order statistics are not sufficient for joint or independent identification of the channel and the carrier frequency offset. First, to illustrate this point we consider a simple noiseless 4-tap channel with impulse response $\boldsymbol{h}^{(1)} = [h_0\ h_1\ h_2\ h_3] = [1\ 2\ 3\ 4]$ with oversampling factor $L = 2$ and frequency offset $\theta_1 = \pi/2$. Define a second channel $\boldsymbol{h}^{(2)} = [h_0\ h_1 e^{j\theta}\ h_2 e^{j2\theta}\ h_3 e^{j3\theta}] = [1, 2j, -3, -4j]$, with no carrier frequency offset ($\theta_2 = 0$). Choosing $m = 2$ results in a signal space of dimension $d = 3$. From (3) and (4) we have

$$\boldsymbol{H}_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \end{bmatrix}, \Theta = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}, \text{ and } \boldsymbol{H}_2 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2j & -4j \end{bmatrix}. \qquad (8)$$

Both systems $\{\boldsymbol{h}^{(1)}, \theta_1\}$ and $\{\boldsymbol{h}^{(2)}, 0\}$ have an identical output autocorrelation $\boldsymbol{R_x}(k) = \mathrm{E}[\boldsymbol{x}(l)\boldsymbol{x}^H(l-k)]$ given by

$$\boldsymbol{R_x}(k) = \begin{bmatrix} 0 & 4j \\ 0 & 0 \end{bmatrix}\delta(k+2) + \begin{bmatrix} -3 & -14j \\ 0 & -8 \end{bmatrix}\delta(k+1) +$$
$$\begin{bmatrix} 10 & 6j \\ -6j & 20 \end{bmatrix}\delta(k) + \begin{bmatrix} -3 & 0 \\ 14j & -8 \end{bmatrix}\delta(k-1) + \begin{bmatrix} 0 & 0 \\ -4j & 0 \end{bmatrix}\delta(k-2), \qquad (9)$$

where $\delta(k)$ is the Kronecker delta function. This example suggests the following theorem.

Theorem 1: Second-order statistics are insufficient for independent or joint identification of the channel and the carrier frequency offset.

Proof: See Appendix A.

We can define an *impairment function* incorporating the effects of the channel and the offset which is identical for both systems, $i = 1$ and $i = 2$:

$$\boldsymbol{g}(\boldsymbol{h}^{(i)},\theta) = \begin{bmatrix} h_0^{(i)} & h_1^{(i)}e^{j\theta_i} & h_2^{(i)}e^{j2\theta_i} & h_3^{(i)}e^{j3\theta_i} \end{bmatrix} = [1 2j -3 -4j]. \qquad (10)$$

In terms of our model, observe that the following matrix product is identical for both systems:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-j\theta_i} \end{bmatrix}\begin{bmatrix} h_0^{(i)} & h_3^{(i)} & 0 \\ 0 & h_2^{(i)} & h_4^{(i)} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j2\theta_i} & 0 \\ 0 & 0 & e^{-j4\theta_i} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2j & -4j \end{bmatrix}. \qquad (11)$$

This matrix product is generalized in the following lemma.

Lemma: Second-order statistics, specifically $\boldsymbol{R_x}(0)$ and $\boldsymbol{R_x}(1)$, are sufficient to identify (to arbitrary phase) the *impairment matrix* $\boldsymbol{G}(\boldsymbol{H},\theta) = \Theta\boldsymbol{H}\Omega^H$, if and only if $\boldsymbol{H}$ is full rank, where

$$\Omega = \mathrm{diag}[1, e^{-j\omega}, e^{-j2\omega}, \ldots e^{-j(d-1)\omega}]. \qquad (12)$$

Proof: See Appendix B.

SOS cannot identify $\boldsymbol{H}$ and θ independently, only the product matrix $\boldsymbol{G}(\boldsymbol{H},\theta) = \Theta\boldsymbol{H}\Omega^H$. Obviously, we can determine $\boldsymbol{H}$ from $\boldsymbol{G}$ if θ were known, because θ determines Θ and Ω, which are invertible. Less obvious is the fact that we can determine θ from $\boldsymbol{G}$ if $\boldsymbol{H}$ were known; see section V. These observations and the lemma lead to the following theorem.

Theorem 2: (a) Second-order statistics are sufficient for channel identification if and only if the carrier frequency offset is known and $\boldsymbol{H}$ is full rank; (b) they are sufficient for carrier recovery if the channel is known and $\boldsymbol{H}$ is full rank.

Note that part (b) is true in the symbol-spaced case as well, except for the notable case when $h_i \propto \delta(i)$. We also state an additional corollary that will be useful later.

Corollary: Higher-order statistics are required to factor the impairment matrix $\boldsymbol{G}(\boldsymbol{H},\theta) = \Theta\boldsymbol{H}\Omega^H$.

Observe that the impairment matrix $\boldsymbol{G}$ (for any θ) has the same Toeplitz-like structure as $\boldsymbol{H}$, as defined by (4). Intuitively, it should therefore not be possible by inspection to distinguish between a given $\boldsymbol{G}_1 = \Theta\boldsymbol{H}_1\Omega^H$ and a second channel $\boldsymbol{G}_2 = \boldsymbol{H}_2$

with no offset. This is just as it was in our previous scalar example. Therefore, in a second-order statistical framework, the tasks of channel estimation and carrier recovery are necessarily coupled. To estimate the channel requires knowledge of the frequency offset; and conversely, to estimate the offset requires some knowledge of the channel. To decouple these tasks necessarily involves the use of higher-order statistics, and requires that the channel input be non-Gaussian.

Even though we cannot explicitly identify the channel, we can still equalize it. Observe that if $\boldsymbol{H}$ is full rank, so is $\boldsymbol{G}$. Therefore, following [6], we can use the Moore-Penrose pseudo-inverse of the impairment matrix to form

$$\begin{aligned}\boldsymbol{z}(k) = \hat{\boldsymbol{G}}^{\dagger}\boldsymbol{x}(k) &= e^{j\beta}\Omega\boldsymbol{H}^{\dagger}\Theta^{H}(e^{jkL\theta}\Theta\boldsymbol{H}\boldsymbol{s}(k) + \boldsymbol{n}(k)) \\ &= e^{j(\beta+kL\theta)}\Omega\hat{\boldsymbol{s}}(k).\end{aligned} \tag{13}$$

where β is the arbitrary phase ambiguity in our estimate of $\boldsymbol{G}$. Then, reverting to scalar notation, we see that this equalizer produces a spinning estimate of the transmitted sequence:

$$z(k) = e^{j(\beta + kL\theta)}\hat{s}(k). \tag{14}$$

Unfortunately, because $s(k)$ is white, the second-order statistics of $z(k)$ contain no information about θ. Therefore, any subsequent carrier recovery requires higher-order statistics, as stated below.

> **Theorem 3**: (a) Second-order statistics are sufficient for channel equalization, independent of carrier recovery, if and only if $\boldsymbol{H}$ is full rank; (b) Second-order statistics are insufficient for joint equalization and carrier-recovery.

In Fig. 2 we present a pictorial summary of our results.

IV. The Effect of Carrier Frequency Offset on the TXK Algorithm

In this section we analyze the effect of carrier frequency offset on the TXK algorithm. We show that TXK actually estimates the impairment function rather than the channel.We then develop a modification that corrects this estimate.

A. Analysis of TXK Algorithm with Carrier Frequency Offset

We now briefly review the TXK algorithm found in [6,11] and examine the effect of frequency offset at each step. Throughout this section, we denote quantities for the case with ideal carrier recovery (no offset) with a tilde (~) and relate them to the quantities computed by TXK with offset.

1. Choose m. L is a known system parameter, and usually we can upper bound the number of taps n in the unknown channel. We want to choose $m > d$ so that we can separate the signal and noise subspaces. With knowledge of L and an upper bound on n, choose m sufficiently large based on (7).

2. Estimate $\boldsymbol{R_x}(0)$ and $\boldsymbol{R_x}(1)$ by time averaging:

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}}(k) = \frac{1}{N}\sum_{i=0}^{N-1}\boldsymbol{x}(i)\boldsymbol{x}^{H}(i-k). \tag{15}$$

   Relating the values with offset to the case with no offset, we have $\boldsymbol{R_x}(k) = e^{jkL\theta}\Theta\tilde{\boldsymbol{R}}_{\boldsymbol{x}}(k)\Theta^{H}$.

3. From $\hat{\boldsymbol{R}}_{\boldsymbol{x}}(0)$, estimate the noise covariance $\sigma^2\boldsymbol{I}$ and the dimension d of the signal space. Recall that the singular value decomposition (SVD) of $\boldsymbol{R_x}(0)$ has the following form:

$$\boldsymbol{U}^{H}\boldsymbol{R}_{x}(0)\boldsymbol{U} = \text{diag}\left[\lambda_1 + \sigma^2, \dots\lambda_d + \sigma^2, \sigma^2, \dots\sigma^2\right], \tag{16}$$

   where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0$. We can relate the unitary matrix $\boldsymbol{U}$ to the ideal case as $\boldsymbol{U} = \Theta\tilde{\boldsymbol{U}}$. The singular values for the two cases are identical: $\lambda_i = \tilde{\lambda}_i \forall i$ and $\sigma = \tilde{\sigma}$.

4. Compute the SVD of $\boldsymbol{R}_0 = \hat{\boldsymbol{R}}_x(0) - \hat{\sigma}^2\boldsymbol{I}$, and form $\boldsymbol{U}_s$, a matrix whose columns are the singular vectors associated with the d largest singular values (i.e. they span the signal space), and Σ, a diagonal matrix of the positive square roots of the d largest singular values. The carrier offset does not affect the noise subspace so we have $\boldsymbol{U}_s = \Theta\tilde{\boldsymbol{U}}_{\boldsymbol{s}}$.

5. Define $\boldsymbol{F} = \Sigma^{-1}\boldsymbol{U_s}^{H}$, and compute the SVD of $\boldsymbol{R} = \boldsymbol{F}(\hat{\boldsymbol{R}}_x(1) - \hat{\boldsymbol{R}}_{\boldsymbol{n}}(1))\boldsymbol{F}^{H}$, where $\hat{\boldsymbol{R}}_{\boldsymbol{n}}(1) = \hat{\sigma}^2\boldsymbol{J}^{L}$. Let $\boldsymbol{y}_d$ and $\boldsymbol{z}_d$ denote the left and right singular vectors corresponding to the smallest singular value. Again, comparing to the ideal case we have $\boldsymbol{F} = \tilde{\boldsymbol{F}}\Theta^{H}$ and $\boldsymbol{R} = e^{jL\theta}\tilde{\boldsymbol{R}}$.

6. Let $\boldsymbol{V} = [\boldsymbol{y}_d,\ \boldsymbol{R}\boldsymbol{y}_d,\ \dots\ \boldsymbol{R}^{(d-1)}\boldsymbol{y}_d]$ or $\boldsymbol{V} = [(\boldsymbol{R}^{\dagger})^{(d-1)}\boldsymbol{z}_d,\ (\boldsymbol{R}^{\dagger})^{(d-2)}\boldsymbol{z}_d,\ \dots\ \boldsymbol{z}_d]$, and compute the matrix $\boldsymbol{U_s}\boldsymbol{S}\boldsymbol{V}$. Thus t $\boldsymbol{V} = \tilde{\boldsymbol{V}}\Omega^{H}$, and thus

$$\boldsymbol{U}_s\Sigma\boldsymbol{V} = \Theta\tilde{\boldsymbol{U}}_s\tilde{\Sigma}\tilde{\boldsymbol{V}}\Omega^{H} = e^{j\alpha}\Theta\hat{\boldsymbol{H}}\Omega^{H} = \hat{\boldsymbol{G}}, \tag{17}$$

   the impairment matrix. The arbitrary phase factor α is due to the fact that the singular value decompositions in steps 3, 4, and 5 are not unique. The singular vectors are specified only to some arbitrary rotation.

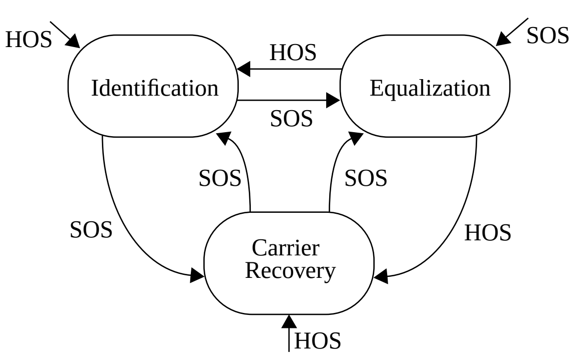

Fig. 2. The relationship among identification, equalization, and carrier recovery.

B. Modified TXK using Decision-Directed Carrier Recovery

We now propose a simple modification to the TXK algorithm to account for frequency offset. The proposed technique uses a decision-directed carrier recovery loop, and thus relies on the discrete nature of the constellation. We have already observed from equations (13) and (14) that a TXK-based equalizer can recover a spinning estimate $z(k)$ of the transmitted sequence. As shown in Fig. 3, a conventional decision-directed phase-locked loop operating on $z(k)$ can generate an accurate estimate of the phase offset θ. The TXK algorithm also produces an estimate of the impairment matrix $\boldsymbol{G}$. As shown in Fig. 3, we can correct the channel estimate using

$$\hat{\boldsymbol{H}} = \Theta^H \hat{\boldsymbol{G}} \Omega . \tag{18}$$

V. Carrier Recovery using Only SOS

We now present a SOS-based method for recovering the carrier offset when the channel is known and $\boldsymbol{H}$ is full rank. Such a method is promised to exist from Theorem 2-b. As shown above we can use the TXK algorithm to estimate the channel impairment matrix $\boldsymbol{G}$ with purely second-order statistics. Compare the SVD of the channel impairment matrix, $\boldsymbol{G} = \boldsymbol{U_G S V_G}^H = (\Theta \boldsymbol{U})\boldsymbol{S}(\boldsymbol{V}^H \Omega^H)$, with that of channel, $\boldsymbol{H} = \boldsymbol{USV}^H$. We can estimate the offset by

$$\hat{\Theta} = e^{j\alpha}(\hat{\boldsymbol{U}}_{\boldsymbol{G}})\boldsymbol{U}^H, \tag{19}$$

or alternatively by

$$\hat{\Omega}^H = e^{j\alpha}\boldsymbol{V}(\hat{\boldsymbol{V}}_{\boldsymbol{G}}^H) . \tag{20}$$

These estimates should be approximately diagonal with the structure of (3) and (12), respectively.

As a second method, we can extract the impairment function $g_k = e^{j\alpha} h_k e^{j\theta k}$ from $\boldsymbol{G}$, where

$$[\boldsymbol{G}_{L\text{-}1,1}\ \boldsymbol{G}_{L\text{-}2,1} \ldots \boldsymbol{G}_{1,1}\ \boldsymbol{G}_{L\text{-}1,2} \ldots \boldsymbol{G}_{1,2} \ldots \boldsymbol{G}_{L\text{-}1,d\text{-}1} \ldots \boldsymbol{G}_{1,d\text{-}1}] = [0 \ldots 0\ g_0\, g_1 \ldots g_{n-1}\ 0 \ldots 0]. \tag{21}$$

With g_k and h_k, θ can be estimated by averaging the difference of the angles produced by g_k / h_k. This works in the symbol-spaced ($L = 1$) case as well, in which the channel matrix would have a Toeplitz structure, again excluding the case $h_i \propto \delta(i)$. This method does not require that the constellation be discrete. Indeed, the channel input could be Gaussian with a continuum of allowable phase transitions.

VI. Simulation Results

The first simulation experiment compares the original TXK algorithm to our modified one in the presence of carrier frequency offset. The channel is the 3-ray, oversampled ($L = 4$) raised cosine from [6]. The autocorrelation estimates were based on 100 symbols, and the amount of carrier frequency offset was 0.75 radian per sample. We ran Monte-Carlo simulations of 100 independent trials of each algorithm. Fig. 4 (a) shows the sample means of channel estimates provided by each algorithm. The original algorithm converges to the impairment function as expected. Our modified algorithm correctly identifies the frequency offset information from HOS and uses it to correct the channel estimate.

We repeated the experiment using a ramp channel. The results are shown in Fig. 4 (b), where the estimated provided by the unmodified algorithm are clearly modulated by a complex sinusoid.

The next simulation verifies the SOS-based carrier recovery algorithm of section V. In this experiment, the transmitted symbols were Gaussian to simulate a highly coded input. The frequency offset θ was 0.08. In each run the autocorrelation was estimated by averaging $N = 500$ symbols in (13). Fig. 5 shows a histogram of 500 Monte Carlo runs. The algorithm appears to work quite well. The estimate in this figure appears to be unbiased with reasonable variance, and this variance can be reduced by increasing N.

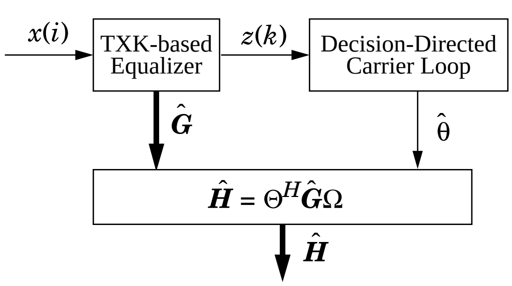

Fig. 3. Modified TXK using a carrier loop.

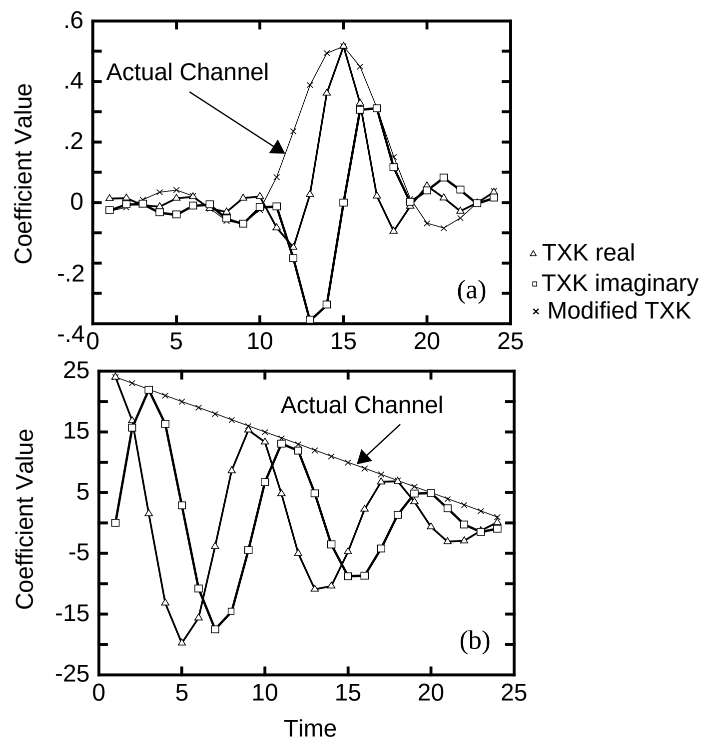

Fig. 4. Comparison of the original and modified TXK algorithms: (a) 3-ray channel from [6] and (b) ramp channel.

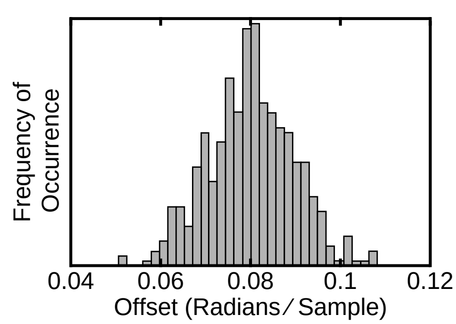

Fig. 5. Histogram of phase estimates.

VII. Conclusions

We have shown that second-order statistics are not sufficient to jointly (or independently) perform blind channel identification and blind carrier recovery. We presented an analysis of the TXK algorithm in the presence of carrier frequency offset and suggested a method to decouple these tasks involving implicit use of HOS. We found that second-order blind carrier recovery is possible, but only with channel knowledge.

We found that second-order blind channel equalization is possible, even in the presence of carrier frequency offset, but any subsequent blind carrier recovery requires HOS. One can therefore expect that, for systems with near-Gaussian input statistics (such as highly coded and shaped systems), the problem of joint blind equalization and carrier recovery will be a difficult one.

Appendix A: Proof of Theorem 1

Choose two arbitrary and distinct carrier frequency offsets θ_1 and θ_2, with corresponding matrices Θ_i and Ω_i as defined in (3) and (12) respectively. Choose an arbitrary channel matrix $\boldsymbol{H}_1$, of full rank with the structure of (6). Let $\boldsymbol{H}_2 = \boldsymbol{P}\boldsymbol{H}_1\boldsymbol{Q}^H$, with

$$\boldsymbol{P} = diag\left[1 \; e^{-j\delta} \; e^{-j2\delta} \; \dots \; e^{-j(m-1)\delta}\right] = \Theta_2^H\Theta_1, \quad (22)$$

$$\boldsymbol{Q} = diag\left[1 \; e^{-jL\delta} \; e^{-j2L\delta} \; \dots \; e^{-j(d-1)L\delta}\right] = \Omega_2^H\Omega_1, \quad (23)$$

and $\delta = \theta_1 - \theta_2$. The output autocorrelation of the second system is given by $\boldsymbol{R}_{\boldsymbol{x}_2}(k) = E[\boldsymbol{x}(l)\,\boldsymbol{x}^H(k-l)] = e^{jL\theta_2 k}\Theta_2\boldsymbol{H}_2\boldsymbol{R}_s(k)\boldsymbol{H}_2^H\Theta_2^H = e^{jL(\theta_1-\delta)k}\Theta_1\,\boldsymbol{H}_1\boldsymbol{Q}^H\boldsymbol{R}_s(k)\boldsymbol{Q}\boldsymbol{H}_1^H\Theta_1^H + \boldsymbol{R}_n(k)$. The autocorrelation of the channel input data from [6,11] is given by

$$R_s(k) = \begin{cases} \boldsymbol{J}^k & k \geq 0 \\ (\boldsymbol{J}^H)^{-k} & k < 0 \end{cases}, \quad (24)$$

where

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \vdots & & \ddots & \end{bmatrix} \quad (25)$$

is a $d \times d$ shifting matrix. Observe that $\boldsymbol{Q}^H\boldsymbol{R}_s(k)\boldsymbol{Q} = e^{jL\delta k}\boldsymbol{R}_s(k)$ because of the special structure of $\boldsymbol{R}_s(k)$. Thus, we have that $\boldsymbol{R}_{\boldsymbol{x}_1}(k) = \boldsymbol{R}_{\boldsymbol{x}_2}(k)$, and the two systems, $\{\boldsymbol{H}_1, \theta_1\}$ and $\{\boldsymbol{H}_2, \theta_2\}$, have identical second-order statistics.

Appendix B: Proof of Lemma

The sufficiency of the rank condition follows closely the proof in [6]. We assume two systems have identical autocorrelation functions at lags 0 and 1. Equating at lag 0 yields $\Theta_1\boldsymbol{H}_1\boldsymbol{H}_1^H\Theta_1^H = \Theta_2\boldsymbol{H}_2\boldsymbol{H}_2^H\Theta_2^H$. This implies $\Theta_2\boldsymbol{H}_2 = \Theta_1\boldsymbol{H}_1\boldsymbol{U}$, where $\boldsymbol{U}$ is some unknown unitary matrix. At lag 1 we have $e^{jL\theta_1}\Theta_1\boldsymbol{H}_1\boldsymbol{J}\boldsymbol{H}_1^H\Theta_1^H = e^{jL\theta_2}\Theta_2\boldsymbol{H}_2\boldsymbol{J}\boldsymbol{H}_2^H\Theta_2^H$. Substituting, we get that $\boldsymbol{U}\boldsymbol{J} = \boldsymbol{J}\boldsymbol{U}e^{jL\delta}$. Using the shifting structure of $\boldsymbol{J}$ and the fact that the columns of $\boldsymbol{U}$ have unit length, we get that

$$\boldsymbol{U} = e^{j\alpha}\boldsymbol{Q}^H = e^{j\alpha}\Omega_1^H\Omega_2, \quad (26)$$

where α is some arbitrary phase. Finally we have that

$$(\Theta\boldsymbol{H}\Omega^H)_2 = e^{j\alpha}(\Theta\boldsymbol{H}\Omega^H)_1. \quad (27)$$

Thus the impairment matrices differ only by the arbitrary phase α. With no carrier drift, (26) reduces to $\boldsymbol{U} = e^{j\alpha}\boldsymbol{I}$, in agreement with [6].

The necessity of the rank condition follows from the observation that $\boldsymbol{G}$ and $\boldsymbol{H}$ have equal rank and the arguments in [7].

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