# Optimization of Full-Rate Full-Diversity Linear Space-Time Codes using the Union Bound

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*Abstract* — **Although many space-time codes sacrifice their rate in order to achieve a high diversity order, such a sacrifice is not necessary. Recent work has reported two instances of a linear space-time code that achieves both a full rate of $\min(t, r)$ and a full diversity order of $tr$ over a $t$-input $r$-output Rayleigh-fading channel [1][2]. We show that such full-rate full-diversity codes are plentiful and can, in fact, be found with probability one by randomly choosing an encoding matrix from an ensemble of matrices with orthonormal columns. However, full rate and full diversity does not guarantee good error-rate performance. Different encoding matrices with the same rate and diversity order can have markedly different error rates. We propose the union bound on word-error rate as an optimization metric and perform constrained optimization to find good space-time codes. For the two-input, two-output Rayleigh channel, we present an optimized code that outperforms the previously reported codes of [1][2] by 1.25 dB at 4 b/s/Hz and a frame-error rate of $10^{-3}$.**

## I. Introduction

We study the design of linear space-time codes [3] with the aim of optimizing performance. Specifically, we aim to minimize the error rate while operating at a given data-rate and signal-to-noise power ratio (SNR) over a Rayleigh-fading Gaussian-noise channel, assuming no other error-control coding is used. Two crucial parameters of a space-time code are the *rate*, defined as the number of complex information symbols conveyed per signaling interval, and the *raw diversity order*, as determined by the rank criterion [4]. The rate measures the amount of redundancy introduced by the space-time code, and the raw diversity order quantifies the effectiveness of the redundancy. Both high rate and high raw diversity order are desirable, since high raw diversity order helps to mitigate fading, and high rate enables the use of small constellations to achieve a given data rate, thus increasing the robustness to noise [4].

When linear codes are used over a $t$-input, $r$-output Rayleigh fading channel, the maximum achievable rate is $\min(t, r)$, as determined by the nominal rank of the channel matrix [3], and the maximum raw diversity order is $tr$ [4]. Although some space-time codes sacrifice rate in order to achieve high diversity order, there is no fundamental reason to trade-off one against the other. For example, algebraic and number-theoretic techniques have been used to develop linear encoders which simultaneously achieve both full rate and full raw diversity order [1][2]. In this paper, we prove that full-rate, full-diversity codes are plentiful. Specifically, we show that when the block length $N$ satisfies $N \geq t$, a randomly chosen $Nt \times N\min(t, r)$ matrix with orthonormal columns achieves a rate of $\min(t, r)$ and a raw diversity order of $tr$ with probability one.

The rate and raw diversity order determine only the asymptotic performance trends. Codes with the same rate and raw diversity order may have markedly different error-rate performance, and it is a nontrivial problem to find, among codes with the same rate and raw diversity order, one that minimizes word-error rate at a given SNR and data rate. This problem was earlier addressed in [5], where random search and its variants were used to obtain encoding matrices with high coding gain. In this paper, we propose the union bound as an optimization metric, since it is known to be a more reliable predictor of performance than the coding gain [6]. Furthermore, we propose the use of constrained gradient descent to perform fast and reliable optimization. Using a combination of random search and gradient descent, we obtain a high-performance encoding matrix for the 4-QAM alphabet.

The results of our optimization indicate that the number-theoretic space-time codes of [1][2] are not optimum in terms of error rate. In fact, for the two-input two-output channel, the two codes outperform a randomly generated encoder by only 0.5 dB at 4 b/s/Hz. In contrast, the code we obtain by optimizing the union bound outperforms the number theoretic codes by 1.25 dB at the same data rate. The union bound is difficult to compute for large channel dimensions. However, the results of optimization for low channel dimensions indicates that some sets of matrices are likely to yield good space-time encoders. Restricting the search to these smaller sets makes optimization easier even for large channel dimensions.

In Section II, we describe the channel model and present some background. In Section III, we show that random linear space-time codes achieve full rate and full diversity order with probability one. In Section IV, we discuss the problem of optimizing performance beyond merely achieving full rate and raw diversity order. Simulation results show the advantage of optimizing the encoding matrix. Section V summarizes the conclusions from this paper.

This research was supported in part by the National Science Foundation, grant no. ITR-0082329, and by the Yamacraw program.

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## II. System Model and Background

We consider a $t$-transmit, $r$-receive antenna static wireless narrowband MIMO channel, modeled as

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{n}_k \,, \tag{1}$$

where $\mathbf{x}_k$ is the $t \times 1$ channel input and $\mathbf{y}_k$ the $r \times 1$ channel output at time $k$. The entries of the $r \times t$ Rayleigh-fading channel matrix $\mathbf{H}$ are independent, circularly symmetric, unit-variance Gaussian random variables. We assume that $\mathbf{H}$ is unknown to the transmitter, but known to the receiver. The noise $\mathbf{n}_k$ is spectrally and temporally white, so that $\mathrm{E}[\mathbf{n}_{k+l}\mathbf{n}_k{}^*] = \delta_l N_0 \mathbf{I}_r$. The SNR $S$ is defined as the ratio of the average received signal energy to the average noise energy at any receive antenna, namely $S = \mathrm{E}[\| \mathbf{x}_k \|^2]/N_0$.

A rate-$K/N$ space-time encoder takes in a $K \times 1$ complex vector $\mathbf{u}$ and generates a $t \times N$ transmit matrix $\mathbf{X}$ from the elements of $\mathbf{u}$. The $i^{\text{th}}$ column of $\mathbf{X}$ is the $t \times 1$ channel input $\mathbf{x}_i$ at time $i$. We restrict attention to *strictly linear* space-time codes which obtain each complex output symbol by some linear combination of the $K$ elements of the input $\mathbf{u}$. To obtain a convenient representation of the encoding process, for any $m \times n$ matrix $\mathbf{A}$, let $\mathbf{a} = \mathrm{vec}(\mathbf{A})$ be the $mn \times 1$ vector formed by stacking the $n$ columns of $\mathbf{A}$ one below the other, and let $\mathrm{mat}(\mathbf{a})$ represent the reverse operation, the values of $m$ and $n$ being implicit in the definition. Now, the encoding process is defined by

$$\mathbf{X} = \mathrm{mat}(\mathbf{L}\mathbf{u}) \,, \tag{2}$$

where the encoding matrix $\mathbf{L}$ completely specifies the code. The restriction to strictly linear codes instead of the more general *linear dispersion* codes [3] does not lead to significant loss in achievable performance [5].

For every code, if there is a discrete alphabet $\mathcal{U}$ of all possible $K \times 1$ input vectors, there is a corresponding discrete alphabet $\mathcal{X}$ of all possible $t \times N$ transmit matrices. The *raw transmit diversity order* of the code is defined as

$$d_t = \min_{\mathbf{X} \neq \mathbf{X}' \in \mathcal{X}} \mathrm{rank}(\mathbf{X} - \mathbf{X}') \,. \tag{3}$$

Defining the difference alphabet $\mathcal{D} = \{\mathbf{d} = \mathbf{u} - \mathbf{u}' \mid \mathbf{u} \neq \mathbf{u}' \in \mathcal{U}\}$, we notice that if $\mathbf{X}$ and $\mathbf{X}'$ are two different code matrices, then $\mathbf{X} - \mathbf{X}' = \mathrm{mat}(\mathbf{L}(\mathbf{u} - \mathbf{u}')) = \mathrm{mat}(\mathbf{L}\mathbf{d})$ for some $\mathbf{d} \in \mathcal{D}$. Using this fact, the transmit diversity order of (3) can be rewritten as:

$$d_t = \min_{\mathbf{d} \in \mathcal{D}} \mathrm{rank}(\mathrm{mat}(\mathbf{L}\mathbf{d})) \,. \tag{4}$$

Clearly, $d_t \leq \min(t, N)$. If the transmitter uses a space-time code with transmit diversity order $d_t$ and the receiver does ML decoding, the total raw diversity order is $d = d_t r$ [4]. In order to attain the maximum possible raw diversity order of $tr$, the code length must be at least as great as the number of transmit antennas, or $N \geq t$.

## III. Full-rate, Full-diversity Codes are Plentiful

In this section, we use probabilistic arguments to show that FRFD codes are easy to find. Specifically, suppose one is given the values of $N$, $t$ and $K$ satisfying $K \leq Nt$. Let $\mathcal{L}$ denote the ensemble of all $Nt \times K$ matrices with orthonormal columns. We will show that for any discrete input alphabet $\mathcal{U}$, a matrix $\mathbf{L}$ drawn randomly from $\mathcal{L}$ achieves transmit diversity order $d_t = \min(t, N)$ with probability one. The following lemma will prove useful.

**Lemma 1.** For any $\mathbf{d} \neq \mathbf{0}$, if $\mathbf{L}$ is chosen uniformly from $\mathcal{L}$, the $t \times N$ matrix $\mathbf{V} = \mathrm{mat}(\mathbf{L}\mathbf{d})$ has full rank $\min(t, N)$ with probability one.

**Proof:** Since $\mathbf{L}$ is random, so is the vector $\mathbf{v} = \mathbf{L}\mathbf{d}$. Since $\mathbf{L}^*\mathbf{L} = \mathbf{I}$ for all $\mathbf{L} \in \mathcal{L}$, we have $\| \mathbf{v} \|^2 = \mathbf{d}^*\mathbf{L}^*\mathbf{L}\mathbf{d} = \mathbf{d}^*\mathbf{d} = \| \mathbf{d} \|^2$. Denote the $Nt$-dimensional complex sphere of radius $\rho$ by $\mathcal{S}_\rho$. The random vector $\mathbf{v}$ always lies on $\mathcal{S}_{\|\mathbf{d}\|}$. Further, for any unitary matrix $\Theta$, the random matrix $\Theta\mathbf{L}$ has the same uniform distribution as $\mathbf{L}$, therefore the vector $\Theta\mathbf{v}$ has the same distribution as $\mathbf{v}$. In other words, the pdf of $\mathbf{v}$ is invariant to all rotations, leading to the fact that $\mathbf{v}$ is uniformly distributed on $\mathcal{S}_{\|\mathbf{d}\|}$. Consequently, the matrix $\mathbf{V} = \mathrm{mat}(\mathbf{v})$ is uniformly distributed over the set of all $t \times N$ matrices whose elements lie on $\mathcal{S}_{\|\mathbf{d}\|}$.

Now, let $\mathbf{G}$ be a $t \times N$ Rayleigh-fading matrix. The random vector $\mathrm{vec}(\mathbf{G})/\| \mathbf{G} \|_{\mathcal{F}}$ has elements that are uniformly distributed on $\mathcal{S}_1$ [7]. Therefore, the matrix $\mathbf{R} = \mathbf{G}/\| \mathbf{G} \|_{\mathcal{F}}$ is uniformly distributed over the set of all $t \times N$ matrices whose elements lie on $\mathcal{S}_1$. Further, since $\mathbf{G}$ is full-rank with probability one, $\mathbf{R}$ is also full-rank with probability one. Comparing the random variables $\mathbf{R}$ and $\mathbf{V}$, we see that $\mathbf{V}$ has the same distribution as $\| \mathbf{d} \| \mathbf{R}$. Therefore,

$$\Pr[\mathbf{V} \text{ has full rank}] = \Pr[\mathbf{R} \text{ has full rank}] = 1. \tag{5}$$

We can now prove the main theorem of this section.

**Theorem 1.** For any countable input alphabet $\mathcal{U}$, an encoding matrix drawn uniformly from $\mathcal{L}$ achieves a transmit diversity order of $\min(t, N)$ with probability one.

**Proof:** A space-time code with encoding matrix $\mathbf{L}$ achieves transmit diversity $d_t = \min(t, N)$ if and only if $\mathrm{mat}(\mathbf{L}\mathbf{d})$ has full rank for all $\mathbf{d} \in \mathcal{D}$. The probability that $d_t = \min(t, N)$ is

$$\Pr\{ \bigcap_{\mathbf{d} \in \mathcal{D}} F_{\mathbf{d}} \} \,, \tag{6}$$

where $F_{\mathbf{d}}$ is the event that $\mathrm{mat}(\mathbf{L}\mathbf{d})$ has full rank. Since the input alphabet is countable, so is the difference alphabet $\mathcal{D}$. Therefore, the intersection in (6) is taken over a countable set. From Lemma 1, $F_{\mathbf{d}}$ is a set of probability one for any non-zero $\mathbf{d} \in \mathcal{D}$. It is well-known that the intersection of countably many probability one events also has probability one. Therefore, the probability that $d_t = \min(t, N)$ reduces to unity, proving the theorem.

To achieve full rate we need $K = N\min(t, r)$. Further, to achieve full raw diversity order, it is necessary to make $d_t = t$ by choosing $N \geq t$. Thus, for $N \geq t$ and any countable input alphabet, a randomly chosen $Nt \times N\min(t, r)$ encoding matrix with orthonormal columns achieves both full rate and full diversity with probability one, according to Theorem 1. In particular, one can let $\mathcal{U} = \mathcal{C}_K$, the set of all $K$-dimensional vectors of complex numbers whose real and imaginary parts are integers. This ensures full diversity over all finite QAM alphabets, since these are a subset of $\mathcal{C}_K$.

## IV. Optimization of the Encoding Matrix

The main result of the previous section is that FRFD codes are very common. However, rate and raw diversity order only determine asymptotic trends and not the exact performance. Codes with the same rate and raw diversity order can have drastically different error-rate performance. A comprehensive design objective should be to find, from the family of space-time encoding matrices of a given dimension, at least one specific matrix which minimizes the error rate while transmitting at a given data rate at a given SNR. The error rate itself is not amenable to analysis. Instead, one can use some related quantity, like the coding gain or the union bound on error rate, as metrics for the optimization. In [5], coding gain was used as the optimization metric, and numerical optimization was performed for some instances of $t$, $N$, $K$, $r$ and $\mathcal{U}$. It has been established that the union bound reflects actual performance more closely than the coding gain [6]. In the following we propose a combination of random search and gradient descent to find encoding matrices which have a low union bound, given the data rate and SNR.

The union bound $f(\mathbf{L})$ on the word error rate is a smooth, continuous function of the encoding matrix $\mathbf{L}$, namely [6]:

$$f(\mathbf{L}) = \frac{1}{2}\sum_{\mathbf{d}\in\mathcal{D}}\prod_{i=1}^{\min(t,N)}\left(1+\frac{|\lambda_i|^2}{4N_0}\right)^{-r}, \tag{7}$$

where $\{\lambda_i\}$ are the singular values of $\mathrm{mat}(\mathbf{Ld})$. Since the summation in (7) is over the difference alphabet $\mathcal{D}$, it is necessary to keep the input alphabet small in order to maintain low complexity. We chose the input alphabet $\mathcal{U}$ to be the set of all $K \times 1$ vectors whose elements are drawn from a unit-energy 4-QAM alphabet. Optimization for larger QAM constellations (and for large values of $K$) becomes difficult primarily due to the difficulty in computing the union bound.

In order to fix the transmit energy, we constrain the $Nt \times K$ encoding matrix $\mathbf{L}$ to have a squared Frobenius norm equal to $K$, so that the total transmit energy per signaling interval is $E = K/N$. In terms of the SNR $S = E/N_0$, (7) reduces to:

$$f(\mathbf{L}) = \frac{1}{2}\sum_{\mathbf{d}\in\mathcal{D}}\prod_{i=1}^{\min(t,N)}\left(1+\frac{NS}{4K}|\lambda_i|^2\right)^{-r}, \tag{8}$$

which clearly shows the dependence of $f(\mathbf{L})$ on SNR. To get meaningful results, the SNR at which optimization is done must be chosen carefully. In what follows, for a given $N$ and $t$, we first simulated a random code and chose the SNR of optimization to be the point at which the frame-error rate was around $10^{-3}$.

For the optimization results in this paper, we fix $K = Nt$, giving a rate of $t$. Let $\mathcal{F}$ denote the set of all $Nt \times Nt$ matrices with squared Frobenius norm is $Nt$. Further, let $\mathcal{G}$ be the set of all $Nt \times Nt$ matrices with unit-norm columns. Finally, let $\mathcal{L}$ denote, as before, the set of all $Nt \times Nt$ matrices with orthonormal columns. Clearly, $\mathcal{L} \subset \mathcal{G} \subset \mathcal{F}$. The optimization problem amounts to a search for a matrix $\mathbf{L}$ with low $f(\mathbf{L})$ among all $\mathbf{L} \in \mathcal{F}$. However, we heuristically find that the smaller sets $\mathcal{G}$ and $\mathcal{L}$ yield matrices with low union bound faster, and can be used as constraint sets for fast optimization.

We used a combination of random search and constrained gradient descent to perform the optimization. First, a random search was performed over one of the constraint sets until an encoding matrix $\mathbf{L}_0$ with low union bound was found. Then constrained gradient descent was performed with the initialization $\mathbf{L}_0$. Let $\mathbf{L}_{i-1}$ be the matrix obtained at the end of the $(i-1)^{\text{th}}$ iteration. The $i^{\text{th}}$ iteration of constrained descent consists of two steps:

Step 1. $\tilde{\mathbf{L}}_i = \mathbf{L}_{i-1} - \mu\nabla f(\mathbf{L}_{i-1})$, for some step-size $\mu$.

Step 2. $\mathbf{L}_i = \tilde{\mathbf{L}}_i$ rounded off to its closest approximation in the constraint set $\mathcal{L}$, $\mathcal{G}$ or $\mathcal{F}$.

As stated earlier, the sets $\mathcal{G}$ and $\mathcal{L}$ contain matrices with low union bound. In particular, among the three constraint sets, the set $\mathcal{L}$ leads to the fastest convergence to a near-optimum $\mathbf{L}$. Interestingly, if constrained gradient descent is performed on some matrix in the set $\mathcal{G}$, the stable result is usually found to be in $\mathcal{L}$. This observation suggests that $\mathcal{L}$ is likely to contain at least one globally optimum encoding matrix. Unfortunately, we have no proof for this proposition.

Some more structure can be observed for the case $N = 2$. Consider the set $\mathcal{K} \subset \mathcal{L}$ of $2t \times 2t$ matrices of the form

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_t & e^{i\pi/4}\mathbf{I}_t \\ \mathbf{Q} & -e^{i\pi/4}\mathbf{Q} \end{bmatrix},$$

for some $t \times t$ unitary matrix $\mathbf{Q}$. In the course of our optimization, we find that the set $\mathcal{K}$ contains near-optimum matrices, and can be used as the constraint set for an efficient search. In particular, for $t = r = N = 2$, the following $4 \times 4$ matrix in $\mathcal{K}$ had the lowest union bound at an SNR of 23 dB among all the matrices found in the course of our search:

$$\mathbf{K}_{2,2,2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & e^{i\pi/4} & 0 \\ 0 & 1 & 0 & e^{i\pi/4} \\ 0.4456 & -0.8952i & -0.4456e^{i\pi/4} & 0.8952e^{i3\pi/4} \\ 0.8952i & -0.4456 & -0.8952e^{i3\pi/4} & 0.4456e^{i\pi/4} \end{bmatrix}.$$

At an SNR of 23 dB, $\mathbf{K}_{2,2,2}$ has a union bound of $6.9 \times 10^{-5}$. In comparison, the encoding matrices presented for the same values of $t$, $r$ and $N$ in [1] and [2] have union bounds of $1.43 \times 10^{-4}$ and $1.56 \times 10^{-4}$, respectively.

To demonstrate the value of optimization, we compare the performance of several FRFD space-time codes of length two ($N = 2$) operating over a two-input two-output Rayleigh-fading channel with 4-QAM. Frames consisting of 50 coded blocks (or equivalently 100 signaling intervals) were transmitted, and ML decoding was performed at the receiver using the sphere decoder [8]. We compare four space-time codes: the optimized code $\mathbf{K}_{2,2,2}$ proposed in this paper, the number theoretic (NT) code of [1], the linear complex field (LCF) code of [2] and a randomly generated code in $\mathcal{L}$. The parameters of the NT and LCF codes were as suggested in the respective sources. The frame-error rates achieved by these codes are shown in Fig. 2. At a frame error rate of $10^{-3}$, the FRFD codes of [1] and [2] outperform the random code by only 0.5 dB. Observe from the slope that the random code does achieve full diversity, as predicted by Theorem 1. The optimized code $\mathbf{K}_{2,2,2}$ outperforms the previously reported codes by 1.25 dB.

Optimized codes for some other values of $t$, $r$ and $N$ can be found in a similar manner. For example, near-optimum codes $\mathbf{L}_{3,3,3}$ (for $t = r = 3$, $N = 3$) and $\mathbf{K}_{4,4,2}$ (for $t = r = 4$, $N = 2$) are shown in Fig. 1. The major problem with the optimization approach is that computation of the union bound is infeasible for large values of $Nt$. While restriction to the smaller sets $\mathcal{L}$ and $\mathcal{K}$ does reduce the computational burden somewhat, developing more tractable optimization metrics and more efficient optimization techniques is an open problem.

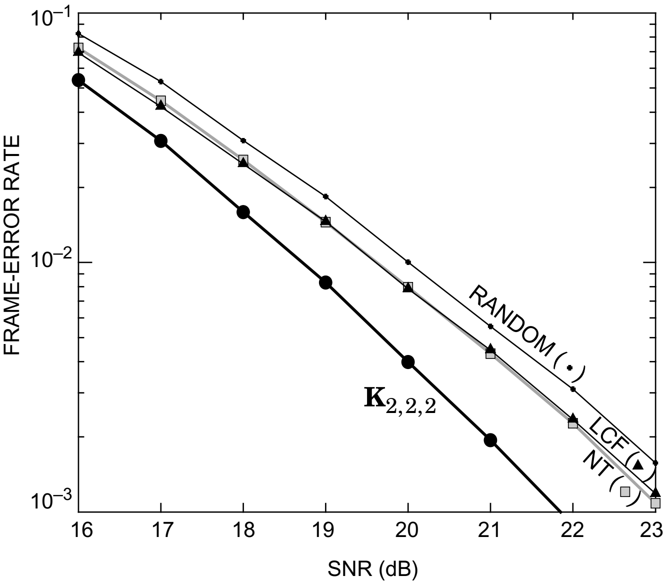


**Fig. 2.** Comparing the proposed space-time code to the NT [1] and LCF [2] codes as well as to a code randomly selected from $\mathcal{L}$.

## V. Conclusion

We showed that full-rate, full-diversity codes are easy to find. In particular, a randomly chosen encoding matrix with orthogonal columns will do. Among FRFD codes, some codes perform better than others, and finding a code which minimizes error rate given the data rate and SNR is a nontrivial optimization problem. We proposed the union bound as an optimization metric, and used a combination of random search and constrained gradient descent to minimize the union bound. The results of our search indicated that encoding matrices with orthonormal columns have near-optimum union bounds. Further, when the code length is $N = 2$, the set $\mathcal{K}$ of $2t \times 2t$ unitary matrices with a special structure was experimentally found to contain near-optimum encoding matrices. Simulation results confirmed the advantage of using optimized encoding matrices. An interesting area of future work is to explain why the heuristically obtained sets $\mathcal{L}$ and $\mathcal{K}$ contain near-optimum matrices.

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$$\mathbf{L}_{3,3,3} =$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}
0.4084e^{-1.2298i} & 0.5453e^{-1.1346i} & 0.3315e^{-0.0493i} & 0.1991e^{-1.4149i} & 0.0842e^{1.3607i} & 1.1160e^{0.3541i} & 0.5723e^{-1.1279i} & 0.5271e^{-1.1611i} & 0.7270e^{0.5640i} \\
0.7206e^{-0.0687i} & 0.6045e^{-0.5798i} & 0.8545e^{0.7839i} & 0.1988e^{1.2204i} & 0.6010e^{-0.0744i} & 0.4126e^{1.1102i} & 0.5536e^{0.0427i} & 0.5542e^{-0.1665i} & 0.4478e^{-1.1425i} \\
0.2019e^{-0.5233i} & 0.5383e^{-0.8568i} & 0.3360e^{0.5376i} & 0.9212e^{-1.0618i} & 0.4494e^{1.2754i} & 0.5721e^{1.1590i} & 0.4964e^{1.5073i} & 0.7679e^{0.8600i} & 0.5855e^{-0.4207i} \\
0.4881e^{1.2558i} & 0.3911e^{1.3722i} & 0.8031e^{0.4948i} & 0.1610e^{1.4392i} & 0.2482e^{-0.2113i} & 0.4303e^{1.0108i} & 1.0836e^{-0.7412i} & 0.6993e^{-0.4208i} & 0.1670e^{0.6923i} \\
0.7517e^{-0.1288i} & 0.3484e^{-0.5263i} & 0.8637e^{-1.0495i} & 0.8123e^{0.9268i} & 0.5438e^{-0.3391i} & 0.3915e^{0.7359i} & 0.1987e^{0.6427i} & 0.3611e^{0.6205i} & 0.5374e^{-0.7823i} \\
0.4573e^{-0.4230i} & 0.9250e^{-1.4284i} & 0.2998e^{1.0788i} & 0.3819e^{1.1240i} & 0.6306e^{-0.0630i} & 0.3587e^{0.8531i} & 0.3833e^{-1.4956i} & 0.4270e^{-0.7812i} & 0.9188e^{-1.0274i} \\
0.2741e^{1.4799i} & 0.3721e^{0.1863i} & 0.4283e^{-0.4484i} & 0.8889e^{-0.6711i} & 1.0906e^{-1.0890i} & 0.3202e^{0.0976i} & 0.3987e^{1.0271i} & 0.1295e^{-1.0401i} & 0.5874e^{-0.8697i} \\
0.3978e^{-0.9336i} & 0.6583e^{-0.8358i} & 0.3337e^{0.9673i} & 0.3963e^{0.4288i} & 0.2317e^{1.4187i} & 0.8217e^{-1.0615i} & 0.7151e^{0.0010i} & 0.9054e^{1.3337i} & 0.2831e^{0.7655i} \\
1.0136e^{1.3700i} & 0.5881e^{-1.1922i} & 0.5210e^{0.4492i} & 0.5418e^{0.5426i} & 0.6569e^{-1.2601i} & 0.1105e^{0.9228i} & 0.2975e^{0.0361i} & 0.4327e^{0.8139i} & 0.5850e^{1.4539i}
\end{bmatrix}$$

$$\mathbf{K}_{4,4,2} = \frac{1}{\sqrt{2}}\begin{bmatrix}
1 & 0 & 0 & 0 & e^{i\pi/4} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & e^{i\pi/4} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & e^{i\pi/4} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & e^{i\pi/4} \\
0.1541e^{-0.0418i} & 0.5637e^{-0.6031i} & 0.5285e^{-1.3616i} & 0.6158e^{-0.4978i} & 0.1541e^{-0.7436i} & 0.5637e^{1.3885i} & 0.5285e^{0.5761i} & 0.6158e^{1.2832i} \\
0.4909e^{0.6967i} & 0.0791e^{0.2708i} & 0.6748e^{-0.0415i} & 0.5453e^{-0.9986i} & 0.4909e^{-1.4821i} & 0.0791e^{0.5146i} & 0.6748e^{0.8269i} & 0.5453e^{0.2132i} \\
0.6386e^{-0.2704i} & 0.5267e^{1.2194i} & 0.0649e^{-1.3155i} & 0.5572e^{-1.4720i} & 0.6386e^{1.0559i} & 0.5267e^{-0.4340i} & 0.0649e^{0.5301i} & 0.5572e^{0.6866i} \\
0.5723e^{-0.4376i} & 0.6313e^{-1.3119i} & 0.5110e^{0.1595i} & 0.1134e^{0.1551i} & 0.5723e^{1.2230i} & 0.6313e^{0.5265i} & 0.5110e^{0.6259i} & 0.1134e^{0.6303i}
\end{bmatrix}$$

**Fig. 1.** Near-optimum encoding matrices $\mathbf{L}_{3,3,3}$ ($t = r = 3$, $N = 3$) and $\mathbf{K}_{4,4,2}$ ($t = r = 4$, $N = 2$).