Approximate Minimum Bit-Error Rate Multiuser Detection

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Abstract — The minimum mean-squared-error (MMSE) linear multiuser detector [1–2] is popular because of its good performance and amenability to adaptive implementation. However, there are circumstances in which the linear detector that minimizes bit-error rate (BER) can significantly outperform the MMSE detector. We propose a low-complexity adaptive algorithm for approximating the minimum-BER linear multiuser detector.

I. Introduction

Within the class of linear multiuser detectors, which includes the conventional matched-filter detector and the decorrelating detector, the minimum-mean-squared-error (MMSE) detector stands out as an attractive choice for two reasons: first, it is amenable to a low-complexity, adaptive, and decentralized implementation [2]; and second, the MMSE detector offers good BER performance [3]. However, as illustrated in this paper, there are circumstances in which the linear detector which chooses its coefficients so as to minimize BER can significantly outperform the MMSE detector.

Other linear detectors have been proposed that can outperform the MMSE detector. The first was the maximum asymptotic-multiuser-efficiency (MAME) linear detector of Lupas and Verdú [4], which minimizes BER in the limit as noise approaches zero. An adaptive algorithm for realizing the MAME detector is not yet available. Adaptive algorithms for realizing the minimum-BER multiuser detector were proposed in [5] and [6], but are either high in complexity or require knowledge of the signature sequence of the user of interest. This paper proposes an adaptive algorithm for approximating the minimum-BER multiuser detector that has low complexity and does not require knowledge of the users signature sequences.

In Sect. II, we present the problem statement. In Sect. III, we discuss exact minimum-BER (EMBER) detection. In Sect. IV, we propose the approximate minimum-BER (AMBER) multiuser detector. In Sect. V, we present numerical results comparing the performance of the AMBER detector to other linear multiuser detectors.

II. Problem Statement

We limit our discussion to a synchronous system, because it captures the essential features of the more general asynchronous problem, and it also clarifies exposition. Consider a synchronous direct-sequence CDMA system with N active users and M chips per baud, and let $\boldsymbol{s}_i$ denote the unit-length M-vector representing the signature sequence of user i. Furthermore, let A_i denote the received amplitude for user i, $b_k^{(i)} \in \{\pm 1\}$ denote the information bit for user i during bit-epoch k, and σ^2 denote the noise PSD. Sampling at the chip rate leads to the following equivalent channel model [3]:

$$\boldsymbol{r}_k = \mathbf{H}\boldsymbol{b}_k + \boldsymbol{n}_k, \tag{1}$$

where the memoryless channel matrix $\mathbf{H} = [\boldsymbol{s}_1\ \boldsymbol{s}_2\ \ldots\ \boldsymbol{s}_N]\mathbf{A}$ has dimension $M \times N$, $\mathbf{A} = \mathrm{diag}(A_i)$, $\boldsymbol{b}_k = [b_k^{(1)} \ldots b_k^{(N)}]^T$, and $\boldsymbol{n}_k$ is white Gaussian noise with PSD $\sigma^2\mathbf{I}$. Without loss of generality, we take user 1 as the user of interest. A decentralized linear multiuser detector for user 1 is then characterized by an M-vector $\boldsymbol{c}$ and the decision rule:

$$\hat{b}_k^{(1)} = \mathrm{sgn}\{\boldsymbol{c}^T\boldsymbol{r}_k\}. \tag{2}$$

The problem is to choose $\boldsymbol{c}$ to minimize the probability that (2) is erroneous, *i.e.*, to minimize the bit-error rate (BER) for user 1. A key contribution of this paper is a low-complexity adaptive algorithm for linear multiuser detectors that closely approximates the minimum-BER solution.

Based on (1), the probability that the decision of (2) is erroneous is:

$$\begin{aligned}\mathrm{BER}_1 &= \Pr[b_k^{(1)}\boldsymbol{c}^T\boldsymbol{r}_k < 0] \\ &= \mathrm{E}\left[\Pr[b_k^{(1)}\boldsymbol{c}^T\mathbf{H}\boldsymbol{b}_k + b_k^{(1)}\boldsymbol{c}^T\boldsymbol{n}_k < 0 \mid \boldsymbol{b}_k]\right] \\ &= \mathrm{E}\left[Q\left(\frac{\boldsymbol{c}^T\mathbf{H}\boldsymbol{b}_k b_k^{(1)}}{\|\boldsymbol{c}\|\sigma}\right)\right],\end{aligned} \tag{3}$$

where the expectations are over the 2^N equally likely binary bit vectors $\boldsymbol{b}_k \in \{\pm 1\}^N$, and where Q is the Gaussian error function. Observe that the product $\boldsymbol{b}_k b_k^{(1)}$ is a binary vector with a one in the first component (corresponding to the user of interest). Let $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \ldots, \boldsymbol{b}^{(L)}$ denote any ordering of the $L = 2^{N-1}$ such distinct vectors. Following [7][8], we define the *signal vectors* by:

$$\boldsymbol{v}^{(i)} = \mathbf{H}\boldsymbol{b}^{(i)},\ i = 1 \ldots L. \tag{4}$$

These $\{\boldsymbol{v}^{(i)}\}$ vectors represent the L possible noiseless channel output vectors given that the k^{th} bit from the desired user is unity, $b_k^{(1)} = 1$. With this definition, (3) simplifies to:

$$\mathrm{BER}_1 = \frac{1}{L}\sum_{i=1}^{L} Q\left(\frac{\boldsymbol{c}^T\boldsymbol{v}^{(i)}}{\|\boldsymbol{c}\|\sigma}\right). \tag{5}$$

Observe that the BER depends on the direction $\boldsymbol{c}/\|\boldsymbol{c}\|$ of $\boldsymbol{c}$ only, and that the norm of $\boldsymbol{c}$ is irrelevant; this is because the receiver decisions are determined by the sign of the detector output only.

In this paper, we make the mild assumption that user i is *linearly detectable*, by which we mean that the signature $\boldsymbol{s}_i$ of user i does not lie within the interference subspace spanned by $\{\boldsymbol{s}_{j \neq i}\}$.

III. Exact Minimum-BER Multiuser Detection

A. The EMBER Detector

Let $\boldsymbol{c}_{EMBER}$ denote a linear multiuser detector that achieves exact minimum-BER (EMBER) performance, minimizing (5). Observe that because (5) depends only on the direction of the detector, $\boldsymbol{c}_{EMBER}$ is not unique: if $\boldsymbol{c}$ minimizes BER, then so does $a\boldsymbol{c}$ for any positive constant a. Unlike the coefficient vector $\boldsymbol{c}_{MMSE} = A_1(\mathbf{HH}^* + \sigma^2\mathbf{I})^{-1}\boldsymbol{s}_1$ that minimizes MSE = E[$(\boldsymbol{c}^T\boldsymbol{r}_k - b_k^{(1)})^2$], there is no closed-form expression for $\boldsymbol{c}_{EMBER}$. However, by setting to zero the gradient of (5) with respect to $\boldsymbol{c}$:

$$\nabla_{\boldsymbol{c}}\mathrm{BER}_1 = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{E}\left[\exp\left(\frac{-(\boldsymbol{c}^T\boldsymbol{v})^2}{2\|\boldsymbol{c}\|^2\sigma^2}\right)\frac{\|\boldsymbol{c}\|^2\boldsymbol{v} - \boldsymbol{c}^T\boldsymbol{v}\boldsymbol{c}}{\|\boldsymbol{c}\|^3}\right] = \mathbf{0}, \tag{6}$$

we find that $\boldsymbol{c}_{EMBER}$ must satisfy:

$$\boldsymbol{c} = af(\boldsymbol{c}) \quad \text{for some } a > 0, \tag{7}$$

where we have introduced the function $f: \mathbb{R}^M \to \mathbb{R}^M$, defined by:

$$f(\boldsymbol{c}) = \mathrm{E}\left[\exp\left(\frac{-(\boldsymbol{c}^T\boldsymbol{v})^2}{2\|\boldsymbol{c}\|^2\sigma^2}\right)\boldsymbol{v}\right]. \tag{8}$$

The expectation in (8) is with respect to $\boldsymbol{v}$ over the $L = 2^{N-1}$ equally likely $\{\boldsymbol{v}^{(i)}\}$ vectors of (4), so that $f(\boldsymbol{c})$ can be expressed as a weighted sum of the $\{\boldsymbol{v}^{(i)}\}$ vectors:

$$f(\boldsymbol{c}) = \frac{1}{L}\sum_{i=1}^{L} e^{-\alpha_i^2/2}\boldsymbol{v}^{(i)}, \tag{9}$$

where $\alpha_i = \boldsymbol{c}^T\boldsymbol{v}^{(i)}/(\|\boldsymbol{c}\|\sigma)$ is a normalized inner product of $\boldsymbol{v}^{(i)}$ with $\boldsymbol{c}$. The fixed-point relationship of (7) characterizes local maxima as well as local minima for the BER cost function, and hence (7) is a necessary but not sufficient test for the global minimum.

Example 1. Consider the simplest nontrivial two-user system described by (1) with $\boldsymbol{s}_1 = [1, 0]^T$, $\boldsymbol{s}_2 = [\rho, \sqrt{1-\rho^2}\,]^T$, normalized correlation $\rho = 0.9$, $\mathrm{SNR}_1 = A_1^2/\sigma^2 = 18$ dB, and $\mathrm{SNR}_2 = 14.6$ dB. In Fig. 1 we present a polar plot of BER_1 versus θ for the unit-norm detector $\boldsymbol{c} = [\cos\theta, \sin\theta]^T$. Superimposed on this plot are the $L = 2$ signal vectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$, depicted by solid lines. Also superimposed are the coefficient vectors of four detectors: the minimum-BER detector at an angle of $\theta = -36.9°$; the MMSE detector at $\theta = -60.2°$; the MF detector at $\theta = 0°$; and the decorrelator at $\theta = -64.2°$. Observe that none of the traditional detectors coincide with the minimum-BER detector. We should point out that the minimum-BER detector is not always colinear with the worst-case signal vector, but rather satisfies $\boldsymbol{c} = af(\boldsymbol{c})$ with $a > 0$ in the general case [see (7) and (9)].

To recover a solution to the fixed-point equation $\boldsymbol{c} = af(\boldsymbol{c})$ with $a > 0$, we propose the *EMBER algorithm*:

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \mu f(\boldsymbol{c}_k), \tag{10}$$

where μ is a small positive step size. Although there may be some solutions to the fixed-point equation $\boldsymbol{c} = af(\boldsymbol{c})$ with $a > 0$ that correspond to local minima and do not globally minimize BER, we hypothesize that if $\boldsymbol{c} = af(\boldsymbol{c})$ and also BER $\le 2^{-N}$, then $\boldsymbol{c}$ minimizes BER. This test is based on the observation that the eye diagram is open when the fixed-point equation is satisfied, and our belief that local minima arise only when certain combinations of interfering user bits are able to close the eye. We thus propose the following strategy for finding the exact minimum-BER linear multiuser detector. First, iterate the deterministic EMBER algorithm of (10) until it converges. If the resulting BER $\le 2^{-N}$, stop. Otherwise, initialize the deterministic EMBER algorithm somewhere else and repeat the process. This is an effective strategy when the initial condition of the EMBER algorithm is chosen carefully (for example, chosen to be the MMSE detector) and when the SNR is sufficiently large that BER $\le 2^{-N}$ is possible.

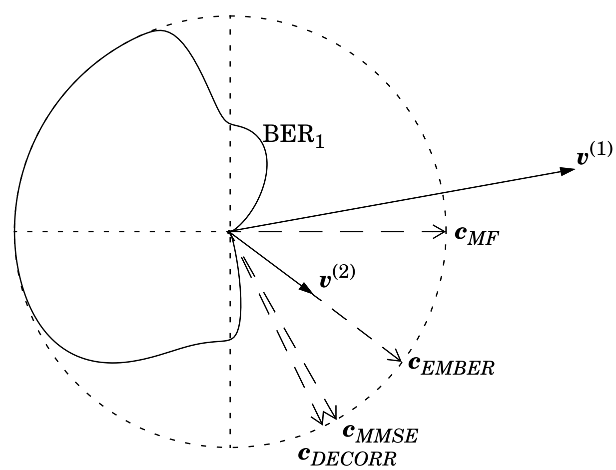

Fig. 1. A polar plot of BER_1 versus θ for a two-user system with correlation $\rho = 0.9$. Superimposed are the signal vectors (scaled by a factor of 0.5) and the MMSE, decorrelating, MF, and minimum-BER detectors.

IV. Approximate Minimum-BER Multiuser Detection

Although the deterministic EMBER algorithm of the previous section is useful for finding the minimum-BER detector of known channels, it is poorly suited for adaptive implementation in time-varying applications. We now propose a stochastic update equation with extremely low complexity and whose ensemble average approximates the EMBER algorithm.

A. The Stochastic AMBER Algorithm

The error function $Q(\alpha)$ is upper bounded and approximated by $\exp(-\alpha^2/2)/(\sqrt{2\pi}\alpha)$ [9], so that $f(\boldsymbol{c})$ of (9) can be approximated by:

$$f(\boldsymbol{c}) \approx \frac{\sqrt{2\pi}}{L} \sum_{i=1}^{L} \alpha_i Q(\alpha_i) \boldsymbol{v}^{(i)} \tag{11}$$

$$\approx \frac{\sqrt{2\pi}}{L} \alpha_{\min} \sum_{i=1}^{L} Q(\alpha_i) \boldsymbol{v}^{(i)} \tag{12}$$

$$= \sqrt{2\pi}\, \alpha_{\min} g(\boldsymbol{c}), \tag{13}$$

where $\alpha_i = \boldsymbol{c}^T \boldsymbol{v}^{(i)} / (\|\boldsymbol{c}\|\, \sigma)$ as in (9), $\alpha_{\min} = \min\{\alpha_i\}$, and where we have introduced the function $g : \mathbb{R}^M \to \mathbb{R}^M$, defined by

$$g(\boldsymbol{c}) = \mathrm{E}\left[Q\left(\frac{\boldsymbol{c}^T \boldsymbol{v}}{\|\boldsymbol{c}\| \sigma} \right) \boldsymbol{v} \right]. \tag{14}$$

The expectation in (14) is with respect to $\boldsymbol{v}$, uniformly distributed over the set of signal vectors (4). The EMBER algorithm of (10) can thus be approximated by

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \mu g(\boldsymbol{c}_k). \tag{15}$$

In analogy to the EMBER fixed-point relationship of (7), we define the AMBER relationship:

$$\boldsymbol{c} = a g(\boldsymbol{c}) \quad \text{for some } a > 0. \tag{16}$$

Let us define an error indicator function I_k that is zero or one, depending on whether an error occurs:

$$I_k = \begin{cases} 0, & \text{if } \operatorname{sgn}\{\boldsymbol{c}^T \boldsymbol{r}_k\} = b_k^{(1)} \\ 1, & \text{if } \operatorname{sgn}\{\boldsymbol{c}^T \boldsymbol{r}_k\} \neq b_k^{(1)} \end{cases}. \tag{17}$$

It follows that $\mathrm{E}[I_k] = \mathrm{BER}_1$. As exploited in (5), the conditional expectation of I_k given $\boldsymbol{v} = b_k^{(1)} \mathbf{H} \boldsymbol{b}_k$ is:

$$\mathrm{E}[I_k \mid \boldsymbol{v}] = Q\left(\frac{\boldsymbol{c}_k^T \boldsymbol{v}}{\|\boldsymbol{c}_k\| \sigma} \right). \tag{18}$$

Therefore, the function $g(\boldsymbol{c})$ can be expressed in terms of this error indicator as follows:

$$\begin{aligned} g(\boldsymbol{c}) &= \mathrm{E}\left[\mathrm{E}[I_k \mid \boldsymbol{v}] \boldsymbol{v} \right] \\ &= \mathrm{E}[I_k \boldsymbol{v}] \\ &= \mathrm{E}[I_k b_k^{(1)} \mathbf{H} \boldsymbol{b}_k] \\ &= \mathrm{E}[I_k b_k^{(1)} (\boldsymbol{r}_k - \boldsymbol{n}_k)]. \end{aligned} \tag{19}$$

Using this result, the deterministic update of (15) can be expressed as:

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \mu \mathrm{E}[I_k b_k^{(1)} (\boldsymbol{r}_k - \boldsymbol{n}_k)] \tag{20}$$

$$\approx \boldsymbol{c}_k + \mu \mathrm{E}[I_k b_k^{(1)} \boldsymbol{r}_k]. \tag{21}$$

The approximation of (21) is valid at high SNR, and is best justified by the good performance of the resulting algorithm, as demonstrated in the numerical results to follow.

We can form a simple stochastic update algorithm by simply removing the expectation in (21):

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \mu I_k b_k^{(1)} \boldsymbol{r}_k. \tag{22}$$

We refer to this stochastic algorithm as the *stochastic AMBER algorithm* for linear multiuser detection, or just AMBER for short.

A closer look at (22) leads to some insightful geometric interpretations of the AMBER algorithm.

- The detector is updated only when an error is made.
- The update term $b_k^{(1)} \boldsymbol{r}_k$ is a noisy estimate of the signal vector: $b_k^{(1)} \boldsymbol{r}_k = b_k^{(1)} (\mathbf{H} \boldsymbol{b}_k + \boldsymbol{n}_k) = \boldsymbol{v} + b_k^{(1)} \boldsymbol{n}_k \approx \boldsymbol{v}$. Hence, when an error is made, $\boldsymbol{c}$ takes a small step in the general direction of the signal vector $\boldsymbol{v}$ that caused the error.
- Averaged over many iterations, $\boldsymbol{c}$ will move towards each $\boldsymbol{v}^{(i)}$ with a frequency proportional to the probability $Q(\alpha_i)$ that $\boldsymbol{v}^{(i)}$ causes an error, where $\alpha_i = \boldsymbol{c}^T \boldsymbol{v}^{(i)} / (\|\boldsymbol{c}\|\, \sigma)$.
- At steady state we expect $\boldsymbol{c} \propto \sum_{i=1}^{L} Q(\alpha_i) \boldsymbol{v}^{(i)}$, which is precisely the AMBER fixed-point relationship of (16), and which closely approximates the minimum-BER fixed-point relationship of (7), namely $\boldsymbol{c} \propto \sum_{i=1}^{L} \exp(-\alpha_i^2/2) \boldsymbol{v}^{(i)}$.

We can gain additional insight into the AMBER algorithm by comparing it to two other well-known adaptive algorithms: the LMS algorithm, which implements the MMSE detector, and the sign-LMS algorithm, a lower-complexity version of the LMS algorithm which approximates the MMSE detector. All three algorithms can be expressed in a similar form:

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k - \mu e_k \boldsymbol{r}_k \qquad \text{(LMS)} \tag{23}$$

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k - \mu \operatorname{sgn}\{e_k\} \boldsymbol{r}_k \qquad \text{(sign-LMS)} \tag{24}$$

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k - \mu I_k \operatorname{sgn}\{e_k\} \boldsymbol{r}_k. \qquad \text{(AMBER)} \tag{25}$$

where $e_k = \boldsymbol{c}^T \boldsymbol{r}_k - b_k^{(1)}$ is the error signal of the MMSE detector, and where we have made use of the identity $I_k b_k^{(1)} = -I_k \operatorname{sgn}\{e_k\}$ to transform (22) to (25). There is a remarkable similarity between the AMBER and sign-LMS algorithms. Simply stated, the AMBER algorithm can be viewed as the sign-LMS algorithm *modified to update only when an error is made*. The sign-LMS was motivated by its low complexity compared to the LMS algorithm, despite its poorer performance. The simple modification for AMBER, on the other hand, provides dramatic performance improvement, without any cost in complexity.

B. Modification of the AMBER Algorithm

Because the AMBER algorithm updates only when an error is made, convergence speed can decrease as the BER decreases. One method for speeding convergence is to modify the AMBER algorithm so that it not only updates when an error is made, but also when an error is almost made. Specifically, we can modify the error indicator function of (17) by introducing a nonnegative threshold $\tau \ge 0$ as follows:

$$I_k = \frac{1}{2}(1 - \mathrm{sgn}[b_k^{(1)} y_k - \tau]). \tag{26}$$

In other words, the modified indicator function is $I_k = 1$ if $b_k^{(1)} y_k \le \tau$ and $I_k = 0$ otherwise. This indicator function reverts back to the original (17) when the threshold τ is zero. Besides speeding convergence, the threshold modification of (26) also allows the AMBER algorithm to be operated in a decision-directed manner, using $\hat{b}_k^{(1)}$ in place of $b_k^{(1)}$ in (22) and (26). Additional methods for speeding convergence are described in [8].

V. Numerical Results

While the MMSE linear multiuser detector generally performs well, the minimum-BER linear detector can perform significantly better. Consider again the two-user system of Example 1 with normalized correlation ρ . In Fig. 2 we illustrate the potential reduction in BER by plotting the ratio $\mathrm{BER}_{\mathrm{EMBER}} / \mathrm{BER}_{\mathrm{MMSE}}$ versus the normalized interference power A_2^2 / A_1^2, assuming $\mathrm{SNR}_1 = 20$ dB. The BER reduction is most pronounced for large correlations and low interference powers.

Next we again consider the two-user system of Example 1, with $\rho = 0.9$ and $A_2^2 / A_1^2 = -4.15$ dB. In Fig. 3 we compare the BER performance of six linear multiuser detectors: the exact minimum-BER detector (satisfying (10)), the approximate minimum-BER detector, the MMSE detector, the matched filter detector, the decorrelator, and the MAME detector. Observe that the exact, approximate minimum-BER, and MAME detectors are indistinguishable. They all outperform the MMSE detector by more than 1 dB at high SNR, and they outperform the matched filter and the decorrelator by a wider margin.

Next we study the performance of these detectors in the presence of strong near-far interference. In Fig. 4 we plot BER_1 versus the normalized interference power A_2^2 / A_1^2, assuming $\rho = 0.9$ and $\mathrm{SNR}_1 = 12$ dB. Interestingly, we see that the MMSE detector approaches minimum-BER performance for extremely low and extremely high interference powers, but not for mod-

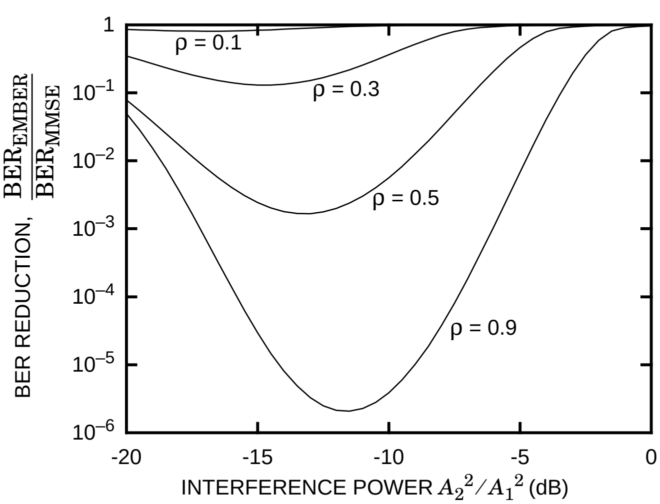

Fig. 2. A plot of $\mathrm{BER}_{\mathrm{EMBER}} / \mathrm{BER}_{\mathrm{MMSE}}$ versus interference power for $\mathrm{SNR}_1 = 20$dB.

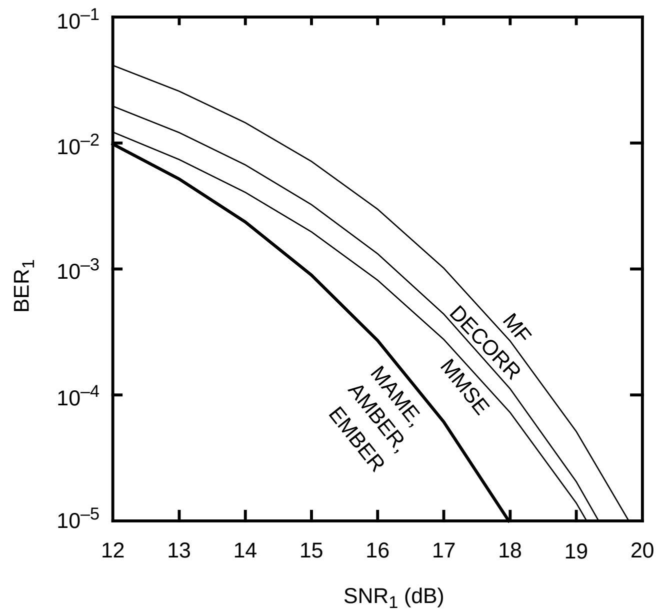

Fig. 3. Theoretical steady-state BER comparison.

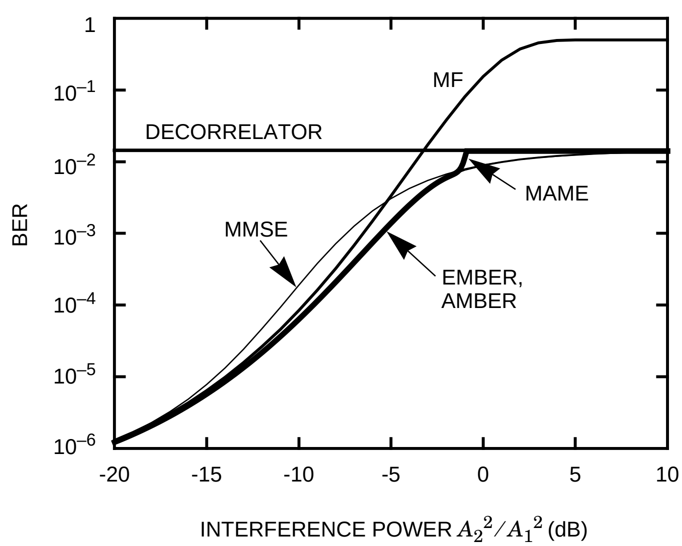

Fig. 4. BER versus normalized interference power.

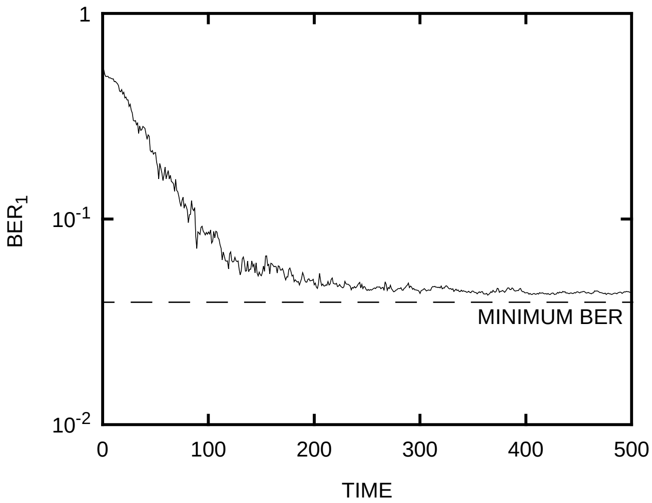

Fig. 5. Learning curve of AMBER in the presence of strong near-far interference and with a closed-eye initialization.

erate interference power. The MAME detector approaches minimum-BER performance at low interference but not at high interference, because for sufficiently high interference, the MAME detector becomes the decorrelator.

The performance of the stochastic AMBER algorithm in the presence of strong near-far interference is examined in Fig. 5, which shows the learning curve of AMBER for a two-user system with $\rho = 0.9$ and $\mathrm{SNR}_1 = 12$ dB, with the interference power exceeding the signal power by 10 dB. The BER was averaged over 100 trials using $\mu = 1 / k$. Before each trial the detector was initialized with the eye closed at $-\boldsymbol{c}_{MMSE}$. Thus, this figure indicates not only that AMBER performs well in a near-far environment but also that it is able to consistently escape a closed-eye situation.

Finally, consider the simple three-user system described by (1) with $\boldsymbol{s}_1 = [1, 0]^T$, $\boldsymbol{s}_2 = [1, 1]^T / \sqrt{2}$, $\boldsymbol{s}_3 = [0, 1]^T$, and SNR = 20 dB for all users. In Fig. 6 we illustrate the performance of the stochastic AMBER algorithm of (22), with parameters $\mu = 0.02$, τ initialized to 0.4, and with τ cut in half at time 200 and again at time 400. Although AMBER can do nothing for user 2, it is of immediate and significant benefit to users 1 and 3.

VI. CONCLUSION

We have proposed the AMBER algorithm for adapting the coefficients of a linear multiuser detector so as to approach the minimum-BER linear detector. When compared to the LMS algorithm, the AMBER algorithm is no more complex and yet can produce a significantly smaller BER. Like the LMS, algorithm, AMBER requires only a training sequence, so that no estimation of the timing or signature waveforms is necessary.

Simulation results reveal no appreciable difference between the steady-state performance of the AMBER algorithm and the optimum minimum-BER linear detector. Furthermore, the AMBER algorithm was shown to be robust to near-far interference and closed-eye initialization, properties that are perhaps not surprising given the close relationship between the AMBER and sign-LMS algorithms.

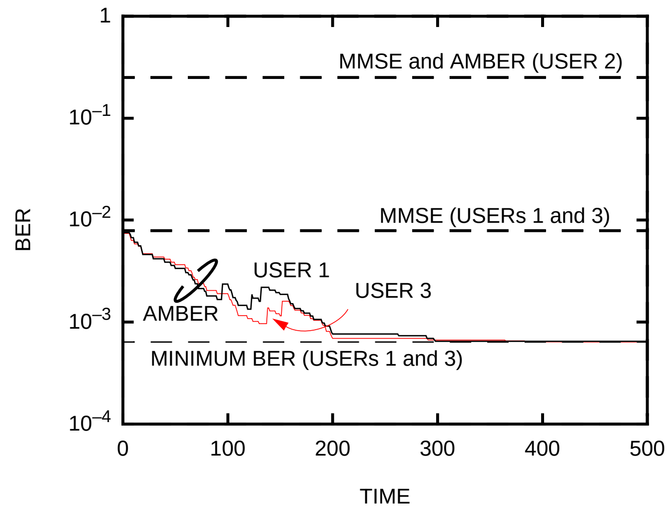

Fig. 6. Learning curves of AMBER for the three-user example.

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