Problem 6.1. Consider a real-valued channel of the form \( r = a_1 h_1 + a_2 h_2 + n \), which is a special case of the MIMO channel considered in class but with only two inputs. Assume \( a_1 \) and \( a_2 \) are independently and uniformly chosen from the binary alphabet \( \{ \pm 1 \} \), and assume that the noise components are real, independent and Gaussian with variance \( \sigma^2 \). This problem compares the jointly optimal (JML) detector, which jointly chooses \( a_1 \) and \( a_2 \) to maximize the conditional pdf \( f(r|a_1, a_2) \), to the individually optimal (IML) detector for the first input, which chooses \( a_1 \) to maximize \( f(r|a_1) \). Let \( y_1 = h_1^T r \), let \( y_2 = h_2^T r \), and let \( \rho = h_2^T h_1 \).

(a) Show that the JML decision for \( a_1 \) can be expressed as:
\[
\hat{a}_1^{\text{JML}} = \text{sign} \left( y_1 + \frac{1}{2} g_0(y_2 - \rho) - \frac{1}{2} g_0(y_2 + \rho) \right),
\]
where we have introduced the nonlinearity \( g_0(x) = |x| \).

(b) Show that the IML decision for \( a_1 \) is also given by the above equation, but with \( g_0(x) = |x| \) replaced by \( g_\sigma(x) = \sigma^2 \log \cosh(x/\sigma^2) \). This nonlinearity is sketched below:

(c) Show that \( g_\sigma(x) \rightarrow |x| \) as \( \sigma \rightarrow 0 \). It follows that \( \hat{a}_1^{\text{IML}} \) and \( \hat{a}_1^{\text{JML}} \) usually agree at high SNR.

(d) Find a value for \( y_1 \) such that \( \hat{a}_1^{\text{IML}} \neq \hat{a}_1^{\text{JML}} \), assuming \( y_2 = 1, \rho = 1 \) and \( \sigma = 1 \).
PROBLEM 5.2. Consider the three-input memoryless channel $H = [h_1, h_2, h_3]$, where $h_1 = [1, 0, 0, 0]^T$, $h_2 = [1, 1, 0, 0]^T$, and $h_3 = [1, 0, 1, 1]^T$. Suppose the inputs are i.i.d. uniformly chosen from $\{\pm 1\}$, and the channel adds real white-Gaussian noise having the identity as an autocorrelation matrix.

(a) Assuming the stack sphere detector is used with the natural ordering, find a numerical value for the probability that the first node extended (besides the root node) is not a part of the actual transmitted path.

(b) Repeat part (a) based on the 2-3-1 ordering (which is the optimal BLAST ordering).

(c) Based on a comparison of (a) and (b), argue qualitatively why the BLAST ordering reduces the complexity of the sphere detector tree search.

PROBLEM 6.3. Consider a 2-by-2 MIMO channel of the form $r = a_1 h_1 + a_2 h_2 + n$, where the components of the noise vector are i.i.d. $CN(0, N_0)$, and where $a_1$ and $a_2$ are independently and uniformly chosen from the 4-QAM alphabet $\{\pm 1 \pm j\}$. Suppose that the channel matrix is:

$$H = [h_1, h_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) Find the weights $w_1^{ZF}$ of the decentralized ZF detector for $a_1$, and find the weights $w_2^{ZF}$ of the decentralized ZF detector for $a_2$.

(b) Find the weights $w_1^{MF}$ of the decentralized MF detector for $a_1$, and find the weights $w_2^{MF}$ of the decentralized MF detector for $a_2$.

(c) Find an equation for the SNR of the decentralized ZF detector for $a_2$.

(d) Find an equation for the error probability of the decentralized ZF detector for $a_1$.

(e) Find an equation for the error probability of the decentralized MF detector for $a_2$.

(f) Find an equation for the single-user-bound (SUB) on error probability for both $a_1$ and $a_2$.

PROBLEM 6.4. A 5-input 10-output MIMO channel transmits an independent 16-QAM symbol from each of the 5 transmit antennas. This question is about the structure of the detection tree.

(a) How many layers are in the detection tree?

(b) How many branches emanate from each non-leaf node in the detection tree?

(c) How many leaf nodes are in the detection tree?

(d) How many total nodes (including the root node and leaf nodes and all in between) are in the detection tree?

(e) How many branches in all are in the detection tree?