Reminder

• HW1 posted, not to be turned in

Lecture: more deterministic DSP

• periodic signals
• sampling & aliasing
• modulation
Remarks:

- Always: can multiply first, filter by $u(f + f_0)$ second
- Often (when $s(t)$ bandlimited to $2f_0$): can replace $u(f + f_0)$ by LPF
  $\Rightarrow$ quad demod
Quad Mod & Demod

**QUADRATURE DEMODULATOR**

\[ s(t) \]
\[ s_I(t) \]
\[ s_Q(t) \]

- Usually equivalent to downconverter

**QUADRATURE MODULATOR**

\[ \sqrt{2} \cos(2\pi f_0 t) \]
\[ -\sqrt{2} \sin(2\pi f_0 t) \]

- Always equivalent to upconverter
Implications of Complex Envelope

Any real signal $s(t)$ can be written uniquely in one of three forms:

C.E. representation: $$s(t) = \sqrt{2} \text{Re}\{\tilde{s}(t)e^{j2\pi f_0 t}\}$$

I &Q representation: $$s(t) = \sqrt{2} s_I(t)\cos(2\pi f_0 t) - \sqrt{2} s_Q(t)\sin(2\pi f_0 t)$$

E&P representation: $$s(t) = e(t)\cos(2\pi f_0 t + \theta(t))$$
Example: \( s(t) = \sqrt{2}\cos(200\pi t) \)

Example 1. w.r.t. a 100 Hz carrier frequency

\( \Rightarrow \) I & Q are 1 and 0, respectively.
\( \Rightarrow \) complex envelope is 1
\( \Rightarrow \) E & P are \( \sqrt{2} \) and 0, respectively.

Example 2. w.r.t. a 99 Hz carrier frequency

\( \Rightarrow \) I & Q are \( \cos(2\pi t) \) and \( \sin(2\pi t) \), respectively.
\( \Rightarrow \) complex envelope is \( e^{j2\pi t} \)
\( \Rightarrow \) E & P are \( \sqrt{2} \) and \( 2\pi t \), respectively.
True or False:

The real envelope $e(t)$ is independent of the carrier frequency.
True or False:

The *real envelope* $e(t)$ is **independent** of the carrier frequency.

True:

$$\tilde{s}(t) = \frac{1}{\sqrt{2}} \left( s(t) + j\hat{s}(t) \right) e^{-j2\pi f_0 t}$$

$$\Rightarrow e(t) = \sqrt{2} |\tilde{s}(t)| = \sqrt{s^2(t) + \hat{s}^2(t)}$$
Pop Quiz: True or False?

The real envelope $e(t)$ has the same energy as the original signal $s(t)$. 
Look in Time Domain

By definition: \[ e(t) = \sqrt{2} |\tilde{s}(t)| \]

⇒ envelope energy is exactly \emph{twice} original
Example: (Real) Envelope of Sinc?

Find the real envelope of \( s(t) = \frac{\sin(\pi t)}{\pi t} \):
Example: (Real) Envelope of Sinc?

Find the real envelope of \( s(t) = \frac{\sin(\pi t)}{\pi t} \):

\[
s(t) = \frac{\sin(\pi t)}{\pi t}
\]

\{(\text{Real})\text{ Envelope of Sinc?}\}
Example: (Real) Envelope of Sinc?

Find the real envelope of \( s(t) = \frac{\sin(\pi t)}{\pi t} \):

Solution:
\[
e(t) = \left| \frac{\sin(\pi t/2)}{\pi t/2} \right|
\]
Example: (Real) Envelope of Sinc?

Find the real envelope of \( s(t) = \frac{\sin(\pi t)}{\pi t} \):

Solution:

\[ e(t) = \left| \frac{\sin(\pi t/2)}{\pi t/2} \right| \]
Periodic Signals and the Fourier Series

\[ x(t) = \sum_{k} a_k e^{jk2\pi t/T} \]

where \( a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk2\pi t/T} dt \)
**Parseval**

**Periodic:**

\[ P = \frac{E_1}{T} = \frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \]

**Nonperiodic:**

\[ E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \]
Example

(a) Find energy $E = ?$

(b) Find power $P = ?$
The FT of an Impulse Train?

\[ p(t) = \sum_k \delta(t - kT) \]
Sampling in 2 Steps

1. Multiply by \( \sum_k \delta(t - kT) \)

2. Convert train of continuous-time impulses to train of discrete-time impulses:

\[
x_c(t) = \sum_k x(kT) \delta(t - kT)
\]

\[
x_k = x(kT)
\]

Compare the Fourier transform of these two signals. They are identical!
Why?
The Impact of Sampling on the F.T.

If the Fourier transform of $x(t)$ looks like this:

Then the F.T. of $x_k = x(kT)$ is:

$$X(e^{j2\pi fT}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X(f - \frac{m}{T}) :$$
Nyquist Sampling Theorem:

Perfect reconstruction is possible when $1/T > 2W$.

How?

Connects dots:

- Linear interpolation $\Rightarrow g(t) = \text{triangle}$.
- Bandlimited interpolation $\Rightarrow g(t) = \text{sinc function}$.
Combine Sampling Theorem with Complex Envelope:

Any passband signal $s(t)$ that is bandlimited to bandwidth $W$:

![Diagram showing passband signal](image)

...can be generated like this:

\[ s_k \xrightarrow{\text{DAC}} S(f) \xrightarrow{\text{UP}} s(t) \]

\[ f_s = W \]

\[ f_0 \]

In other words, any such signal is *uniquely* specified by:

- a carrier frequency $f_0$
- a symbol rate $f_s = W$
- a complex-valued discrete-time sequence $\{s_k\}$.

(How to find it?)
Example: Design Comm System

\[ f(t) \]

\[ s(t) \]

\[ r(t) \]

FCC LIMIT

2.39 GHz

2.41 GHz
Example: Design Comm System

FCC LIMIT

\[ f \]

\[ a_k \]

\[ \tilde{s}(t) \]

\[ DAC \]

20 MHz

\[ UP \]

2.4 GHz

\[ s(t) \]

\[ 2.39 \text{ GHz} \]

\[ 2.41 \text{ GHz} \]

\[ r(t) \]

\[ \tilde{r}(t) \]

\[ DOWN \]

2.4 GHz

\[ ADC \]

20 MHz

\[ r_k \]
Quad Mod & Demod OK

$2.39 \text{ GHz}$  \hspace{1cm}  $2.41 \text{ GHz}$

FCC LIMIT

$\sqrt{2} \cos(2\pi (2.4 \text{ GHz})t)$

$\sqrt{2} \sin(2\pi (2.4 \text{ GHz})t)$

DAC

$10 \text{ MHz}$

ADC

$20 \text{ MHz}$

DAC

$10 \text{ MHz}$

ADC

$20 \text{ MHz}$

LPF

Observed time signal $s(t)$ is divided into $I$ and $Q$.

$20 \text{ MHz}$

$10 \text{ MHz}$

$20 \text{ MHz}$

$20 \text{ MHz}$

$\cos(2\pi (2.4 \text{ GHz})t)$

$\sin(2\pi (2.4 \text{ GHz})t)$

$20 \text{ MHz}$

$10 \text{ MHz}$
Alphabet

- Define $a_k = a_k(I) + j a_k(Q)$ as $k$-th complex-valued “symbol”
- Assume $a_k$ drawn independent/uniform from finite alphabet $\mathcal{A}$ of size $|\mathcal{A}|$
  \[ \Rightarrow R_b = W \log_2 |\mathcal{A}| \]
- Common choices
  - $\mathcal{A} = \{\pm 1\}$ “2-PSK = BPSK”
  - $\mathcal{A} = \{1, j, -1, -j\}$ “4-PSK = QPSK”
  - $\mathcal{A} = \{\pm 1 \pm j\}$ “4-QAM”
PSK Alphabets

$M = 2$

$M = 4$

$M = 8$

$M = 16$

$M = 32$