Lecture 3: Tue Jan 15, 2019

Announce

• HW1 posted, not to be turned in

Lecture: more deterministic DSP

• review I&Q, complex envelopes
• downconversion vs quadrature demodulator
• phase splitter
• Hilbert transform
• periodic signals and the Fourier series
What is FT of Complex Envelope?

Key properties of downconverter:

- invertible = info lossless
  \[ \Rightarrow \text{WLOG.} \]
  - with \( f_0 \) carefully chosen, output is often baseband
    \[ \Rightarrow \text{easier to process, much lower sampling rate} \]

For these reasons: 1st step at receiver is often DOWNCONVERT.
Downconverter: 2 Ways to Implement

1. shift
2. double
3. reject

\[ x(t) \xrightarrow{e^{-j2\pi f_0 t}} 2u(f + f_0) \xrightarrow{\tilde{x}(t)} \]

\[ \tilde{X}(f) = 2u(f + f_0)X(f + f_0) \]

gives eqn in freq domain

\[ x(t) \xrightarrow{2u(f)} \xrightarrow{e^{-j2\pi f_0 t}} \tilde{x}(t) = \]

gives eqn in time domain
Upconverter: Inverse of Downconverter

Upconverter: 

\[ x(t) = \text{Re}\{\tilde{x}(t)e^{j2\pi f_0 t}\} . \]
Downconverter $\approx$ Quad Demodulator

$$s(t) \xrightarrow{2u(f)} e^{-j2\pi f_0 t} \xrightarrow{\times} \tilde{s}(t)$$

$$s(t) \xrightarrow{\cos(2\pi f_0 t)} \xrightarrow{\sin(2\pi f_0 t)} x(t)$$

$$s(t) \xrightarrow{\cos(2\pi f_0 t)} \xrightarrow{\sin(2\pi f_0 t)} y(t)$$
Implications of Complex Envelope

Any real signal $x(t)$ can be written *uniquely* in one of three forms:

- **C.E. representation:** $x(t) = \text{Re}\{\tilde{x}(t)e^{j2\pi f_0 t}\}$
- **I &Q representation:** $x(t) = x_I(t)\cos(2\pi f_0 t) - x_Q(t)\sin(2\pi f_0 t)$
- **E&P representation:** $x(t) = e(t)\cos(2\pi f_0 t + \theta(t))$

How are $e(t) \geq 0$ and $\theta(t) \in (-\pi, \pi]$ (uniquely) defined?
E & P Decomposition

Decompose \( \tilde{x}(t) = e(t)e^{j\theta(t)} \):

\[
e(t) = |\tilde{x}(t)| = \text{envelope of } x(t) \text{ w.r.t. } f_0
\]

\[
\theta(t) = \arg\{\tilde{x}(t)\} = \text{angle of } x(t) \text{ w.r.t. } f_0:
\]

\[
\Rightarrow s(t) = \Re\{e(t)e^{j\theta(t)}e^{j2\pi f_0 t}\}
\]

\[
\Rightarrow \text{Any real signal can be written as } \quad x(t) = e(t)\cos(2\pi f_0 t + \theta(t)).
\]
Pop Quiz: True or False?

The real envelope $e(t)$ has the same energy as the original signal $x(t)$. 
Look in Time Domain

By definition: \( e(t) = |\tilde{x}(t)| \)

⇒ real envelope and complex envelope have the same energy

⇒ real envelope energy is exactly twice original
Pop Quiz: \( s(t) = \cos(200\pi t) \)

Find I&Q components, complex envelope, and envelope & phase
**Pop Quiz:** \( s(t) = \cos(200\pi t) \)

Find I&Q components, complex envelope, and envelope & phase

**Example 1.** w.r.t. a 100 Hz carrier frequency
Pop Quiz: $s(t) = \cos(200\pi t)$

Find I&Q components, complex envelope, and envelope & phase

**Example 1.** w.r.t. a 100 Hz carrier frequency

⇒ I & Q are 1 and 0, respectively.

⇒ complex envelope is 1

⇒ E & P are 1 and 0, respectively.

**Example 2.** w.r.t. a 99 Hz carrier frequency
Pop Quiz: \( s(t) = \cos(200\pi t) \)

Find I&Q components, complex envelope, and envelope & phase

Example 1. w.r.t. a 100 Hz carrier frequency

\[ \Rightarrow I \& Q \text{ are } 1 \text{ and } 0, \text{ respectively.} \]
\[ \Rightarrow \text{complex envelope is } 1 \]
\[ \Rightarrow E \& P \text{ are } 1 \text{ and } 0, \text{ respectively.} \]

Example 2. w.r.t. a 99 Hz carrier frequency

\[ \Rightarrow I \& Q \text{ are } \cos(2\pi t) \text{ and } \sin(2\pi t), \text{ respectively.} \]
\[ \Rightarrow \text{complex envelope is } e^{j2\pi t} \]
\[ \Rightarrow E \& P \text{ are } 1 \text{ and } 2\pi t, \text{ respectively.} \]
Hilbert Transform

The *Hilbert transform* \( \hat{s}(t) \) of a *real* signal \( s(t) \) is defined as the output of a Hilbert transformer when \( s(t) \) is the input:

- \( H(f) \) has complex-conjugate symmetry \( \Rightarrow \hat{x}(t) \) real
- Hilbert transformer does not modify the magnitude spectrum
- Gives a \(-\pi/2\) phase shift at all frequencies.
Example 1

Find the Hilbert transform of $x(t) = \cos(6602\pi t)$
Pop Quiz

(a) In time domain, is impulse response \( h(t) \) real or complex?
(b) What is FT of \( \text{Re}\{h(t)\}\)?
(c) What is FT of \( \text{Im}\{h(t)\}\)?
Phase Splitter

Real-valued input $x(t)$, complex-valued output $x_+(t) = x(t) + j\hat{x}(t)$:

Overall frequency response is $2u(f)$:

Output signal has only nonnegative frequency components

$\Rightarrow$ analytic

$\Rightarrow$ complex
Equations for the Complex Envelope

The complex envelope $\tilde{x}(t)$ of a real signal $x(t)$ with respect to $f_0$ is defined as the output of a downconverter with carrier $f_0$:

$$\tilde{X}(f) = 2u(f + f_0)X(f + f_0)$$

$$\tilde{x}(t) = \left(x(t) + j\hat{x}(t)\right)e^{-j2\pi f_0 t}.$$ 

Properties of the complex envelope:

- unique
- Lossless: can convert complex envelope back to original signal
- Often low-pass $\Rightarrow$ easier to process

For these reasons, the very first step at a receiver is often downconversion.
Pop Quiz: True or False?

The complex envelope $\tilde{x}(t)$ has exactly the same energy as the original $x(t)$. 
Look in Frequency Domain:

\[ X(f) \]

\[ \tilde{X}(f) = 2u(f + f_0)S(f + f_0) \]
Parseval

Nonperiodic: \[ E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \]

Periodic: \[ P = \frac{E_1}{T} = \frac{1}{T} \int_{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \]
Example

(a) Find energy $E = ?$

(b) Find power $P = ?$
The FT of an Impulse Train?

\[ p(t) = \sum_k \delta(t - kT) \]