PROBLEM 5.1. Consider the case of Gray-coded 16-QAM in AWGN, where the received vector \( \mathbf{r} \) is modeled as:

\[
\mathbf{r} = \mathbf{s} + \mathbf{n},
\]

where \( \mathbf{s} \) is equally likely to be any of the \( M = 16 \) signal vectors shown in the 16-QAM constellation diagram here (where \( s_1, s_2 \in \{\pm 1, \pm 3\} \)), and where the two components of the noise vector \( \mathbf{n} \) are i.i.d. \( \mathcal{N}(0, \sigma^2) \). Also shown in the diagram is the mapping from each four-bit message \( \mathbf{m} = [m_0 \ m_1 \ m_2 \ m_3] \) to its corresponding signal vector \( \mathbf{s} \), where \( m_i \in \{0, 1\} \).

(This is called a “Gray” mapping because the messages for neighboring signal vectors differ in only a single bit position.)

A “soft demapper” (or “LLR computer”) is a device at the receiver that takes as an input the received signal \( \mathbf{r} \), and produces as its output four log-likelihood ratios \( \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} \), one for each message bit, where:

\[
\lambda_i = \log \frac{f(\mathbf{r} | m_i = 1)}{f(\mathbf{r} | m_i = 0)}.
\]

(a) Find exact equations for \( \lambda_0 \) through \( \lambda_3 \), expressed as a function of the components of \( \mathbf{r} = [r_1 \ r_2] \) and \( \sigma^2 \).

(b) Compare the exact equations from (a) to the following commonly used approximations:

\[
\begin{align*}
\lambda_0 &\approx 2 \frac{r_1 + 2}{\sigma^2} & \text{for } r_1 < -2 \\
&\approx 2 \frac{r_1 - 2}{\sigma^2} & \text{for } r_1 > 2 \\
\lambda_1 &\approx \frac{2}{\sigma^2} (2 - |r_1|) & \text{for } r_1 \in [-2, 2] \\
\lambda_2 &\approx 2 \frac{r_2 + 2}{\sigma^2} & \text{for } r_2 < -2 \\
&\approx 2 \frac{r_2 - 2}{\sigma^2} & \text{for } r_2 > 2 \\
\lambda_3 &\approx \frac{2}{\sigma^2} (2 - |r_2|) & \text{for } r_2 \in [-2, 2].
\end{align*}
\]

Make the comparison by plotting four graphs, assuming \( \sigma^2 = 0.2 \) in all cases:

- \( \lambda_0 \) vs \( r_1 \) for both cases (exact vs approximate),
- \( \lambda_1 \) vs \( r_1 \) for both cases (exact vs approximate),
- \( \lambda_2 \) vs \( r_2 \) for both cases (exact vs approximate),
- \( \lambda_3 \) vs \( r_2 \) for both cases (exact vs approximate).

(c) Repeat part (a) for the much simpler case of Gray-coded 4-QAM, as shown here: Find exact equations for \( \lambda_0 \) and \( \lambda_1 \), expressed as a function of the components of \( \mathbf{r} \) and \( \sigma^2 \). \textbf{Hint:} They will be very simple!