PROBLEM 2.1. Consider the signals $r(t)$, $s_1(t)$ and $s_2(t)$ sketched below:

Find and carefully sketch $\hat{r}(t)$, the projection of $r(t)$ onto the space spanned by $s_1(t)$ and $s_2(t)$.

PROBLEM 2.2. Let $\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\}$ be an orthonormal basis for a subspace $S$. Let us define three new waveforms $\{f_1(t), f_2(t), f_3(t)\}$ according to:

- $f_1(t) = A_{1,1}\varphi_1(t) + A_{1,2}\varphi_2(t) + A_{1,3}\varphi_3(t)$
- $f_2(t) = A_{2,1}\varphi_1(t) + A_{2,2}\varphi_2(t) + A_{2,3}\varphi_3(t)$
- $f_3(t) = A_{3,1}\varphi_1(t) + A_{3,2}\varphi_2(t) + A_{3,3}\varphi_3(t)$

Under what conditions will $\{f_1(t), f_2(t), f_3(t)\}$ also be an orthonormal basis for $S$? Express your answer in terms of the matrix $A$ containing $A_{i,j}$ in row $i$, column $j$.

PROBLEM 2.3. 

(a) Find the energy of $x(t) = g(t) + g(t - 6)$, where $g(t) = \sin(\pi t/3)/(\pi t/3)$.

(b) Find coefficients $a$ and $b$ so that $a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(c) Find an orthonormal basis for the span of $x = [1, 2, 3, 4]^T$ and $y = [3, 0, 4, 0]^T$. 
PROBLEM 2.4. Let \( S = \text{span}\{g(t), g(t-1), g(t-2), g(t-3)\} \), where \( g(t) = u(t) - u(t-1) \) is a rectangular pulse of unit duration.

(a) Find the dimension \( N = \dim(S) \).
(b) Find the projection \( \hat{s}(t) \) of \( s(t) = e^{-t}u(t) \) onto \( S \).
(c) Sketch \( \hat{s}(t) \) and \( s(t) \) on the same graph.
(d) Calculate the energy of the error signal, \( E_e = \int_{-\infty}^{\infty} |\hat{s}(t) - s(t)|^2 dt \).

PROBLEM 2.5. Sketch the signal space diagram (constellation) for a binary modulation scheme that transmits either \( s_1(t) \) or \( s_2(t) \), where these waveforms are defined in the figure below:

(a) True or False: \( N' = N \), and \( f_i(t) = \phi_i(t) \) for all \( i = 1, \ldots, N \).
(b) If true, prove it.
If false, give a counterexample.

PROBLEM 2.6. Suppose you have a set of real-valued waveforms \( \{s_1(t), s_2(t), \ldots, s_N(t)\} \), and you want to find a basis for the span of their complex envelopes. The obvious approach would be to first downconvert each of the waveforms, and then apply the Gram-Schmidt procedure to the set of complex envelopes. Will we get the same answer if we first apply Gram-Schmidt, and second downconvert?

Let us restate the problem in more precise terms. Let \( \{\phi_1(t), \phi_2(t), \ldots, \phi_N(t)\} \) be the basis found by applying the Gram-Schmidt procedure to a linearly independent set of real-valued waveforms \( \{s_1(t), s_2(t), \ldots, s_N(t)\} \). Let \( \tilde{s}_i(t) \) denote the complex envelope of \( s_i(t) \) with respect to a particular carrier frequency \( f_0 \). Let \( \tilde{\phi}_i(t) \) denote the complex envelope of \( \phi_i(t) \) with respect to the same carrier frequency. Let \( \{f_1(t), f_2(t), \ldots, f_N(t)\} \) be the basis for the span of \( \{\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_N(t)\} \), as found by applying the Gram-Schmidt procedure to \( \{\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_N(t)\} \).

(a) True or False: \( N' = N \), and \( f_i(t) = \phi_i(t) \) for all \( i = 1, \ldots, N \).
(b) If true, prove it.
If false, give a counterexample.

PROBLEM 2.7. Let \( S \) be the linear subspace consisting of all waveforms that are identically zero outside of the interval \( t \in [a, b] \), and that are linear inside the interval \( t \in [a, b] \). An example of one such waveform is illustrated in the figure. Find and carefully sketch an orthonormal basis for the subspace \( S \).
PROBLEM 2.8. Consider the three piecewise-constant waveforms shown on the right. In terms of an orthonormal basis \( \{ \varphi_1(t), \varphi_2(t) \} \) for the subspace \( S = \text{span}\{x(t), y(t), z(t)\} \), we may write \( x(t) = x_1 \varphi_1(t) + x_2 \varphi_2(t) \), \( y(t) = y_1 \varphi_1(t) + y_2 \varphi_2(t) \), and \( z(t) = z_1 \varphi_1(t) + z_2 \varphi_1(t) \). Define \( x = [x_1, x_2]^T \), \( y = [y_1, y_2]^T \), and \( z = [z_1, z_2]^T \). The basis for a subspace is not unique; we will find three different bases for \( S \).

(a) Find and carefully sketch the basis functions \( \varphi_1(t) \) and \( \varphi_2(t) \) when the vectors \( x, y, \) and \( z \) are as shown in (a) below.

(b) Find and carefully sketch the basis functions \( \varphi_1(t) \) and \( \varphi_2(t) \) when the vectors \( x, y, \) and \( z \) are as shown in (b) below.

(c) Find and carefully sketch the basis functions \( \varphi_1(t) \) and \( \varphi_2(t) \) when the vectors \( x, y, \) and \( z \) are as shown in (c) below.

PROBLEM 2.9. Consider a binary modulation scheme defined by \( s_1(t) = \sin(2\pi t)/(2\pi t) \) and \( s_2(t) = \sin(4\pi t)/(4\pi t) \), as sketched below:

Sketch the signal-space diagram (or constellation).

(Hint: You may find it helpful to use the generalized Parseval result.)

PROBLEM 2.10. Sketch the signal-space diagram for the 64-ary modulation scheme in which the transmitter sends a signal of the form \( s(t) = a_1 g_1(t) + a_2 g_2(t) \), where \( g_1(t) = \sin(\pi t)/(\pi t) \) and \( g_2(t) = g_1(t-1) \); the first symbol alphabet is \( a_1 \in \{ \pm 1, \pm 3 \} \), and the second is \( a_2 \in \{ \pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15 \} \).