Before Shannon

Nyquist [1924]: The maximum symbol rate is \( W \) complex symbols/second.
- complex symbol sent as: \( \text{real} \times \cos(\cdot) - \text{imaginary} \times \sin(\cdot) \)

If the symbol alphabet is \( \mathcal{A} \), then each symbol conveys \( \log_2 |\mathcal{A}| \) bits.

- \( \mathcal{A} = \{-1, 1\} \) \( \Rightarrow \) \( R_b = W \cdot \log_2 |\mathcal{A}| \). (BPSK)
- \( \mathcal{A} = \{\pm 1, \pm j\} \) \( \Rightarrow \) \( R_b = 2W \). (4-QAM)
- \( \mathcal{A} = \{1, 2, 3, \ldots\} \) \( \Rightarrow \) \( R_b = \infty \).
- \( \mathcal{A} = \{0, 1\} \) \( \Rightarrow \) \( R_b = \infty \).

Hartley [1928]: With amplitude constraint \( \pm A \), and deterministic resolution \( \pm \Delta \), the maximum bit rate is:

\[
R_b = W \cdot \log_2 \left( \frac{A + \Delta}{\Delta} \right).
\]

Modulation was instantaneous. To fight random noise:
- increase BW, increase power, live with “inevitable” errors.

The Turning Point: Claude Shannon, 1948

The father of information theory and modern communication theory

1948 paper — A (The) Mathematical Theory of Communication
- quantified information; separation theorem; noisy channel theorem

Two forms of the channel capacity, for \( H(f) = \text{constant over bandwidth } W \):
- given \( W, P \), the maximum bit rate is:

\[
C = W \cdot \log_2 \left( 1 + \frac{P}{N_0 W} \right).
\]
- given \( W, R_b \), the minimum bit energy is:

\[
E_b / N_0 = \frac{2^{R_b / W} - 1}{R_b / W}.
\]

The speed limit:
- \( \exists \) a modem achieving bit rate \( R_b \) with no errors \( \iff \) \( R_b < C \).

How to get close?
Source Coding = Data Compression

Associated with each discrete source is an entropy $H$ bits/symbol.
The compression limit:
\[ \exists \text{ a compressor to } B \text{ bits/symbol with } \Pr[\text{err}] \to 0 \text{ as } N \to \infty \iff B > H. \]

Separation of Compression, Coding, and Equalization

The Source-Channel Coding Separation Theorem
- compress source independent of channel
- channel encoder independent of source

There is no “Equalization-Coding” separation theorem
- fighting $H(f)$ and noise together is better than separately, in theory
- One common strategy: separate anyway
  - first, “equalize” to combat effects of $H(f)$
    - decision-feedback equalization
    - Tomlinson precoding
  - second, code for an ideal $[H(f) = 1]$ channel
  - asymptotic penalty is less than 0.6 dB (or 6% of capacity).
- Another strategy: multicarrier modulation

The Coding Problem

After equalization, and after $n$ symbol transmissions with symbol rate $W$:
\[ r = a + w \]

where $\mathbb{E}[\|w\|^2] = nN_0$.

The Problem

Find a set of $2^nR_b/W$ codewords (of length $n$, satisfying $\mathbb{E}[\|a\|^2] < nP/W$) and a decoding rule such that $\Pr[\text{err}] \leq 10^{-6}$.

How many ping pong balls fit inside a beach ball?
\[ 2^nR_b/W = \frac{\text{volume of beach ball}}{\text{volume of ping-pong ball}} = \left(1 + \frac{P}{N_0W}\right)^n. \]

$\Rightarrow$ Again, $R_b = C = W \cdot \log_2 \left(1 + \frac{P}{N_0W}\right)$.

(Requires $n \to \infty$).

In theory, both a regular lattice and a random scattering achieves $C$ as $n \to \infty$. 

The Water Pouring Procedure for Achieving Capacity

1. Create bowl $\beta(f) = \frac{N_0}{2}H(f)$
2. Pour water of volume $P$
3. \[ C = W \cdot \log_2 \left(\frac{A + P/(2W)}{G}\right). \]

Multicarrier Modulation — approximates this ideal by sending independent data across numerous narrow subchannels.

If subchannels are narrow enough, the channel $H(f)$ appears approximately flat, and equalization is not necessary.

Applications — ADSL, trailblazer

Can allocate power and bits optimally to each subchannel.
Good codes are known; good decoders are not. Complexity of minimum-distance decoding grows exponentially in $n$. The practical problem: to *automate* random-like encoder and its decoder.

Two regimes:

- **Power** limited ($R_b / W < 1$ bit per symbol)
  - deep-space probes, satellites, pagers
  - binary codes are sufficient
  - block or convolutional codes + 4-QAM alphabet
- **Bandwidth** limited ($R_b / W > 1$ bits per symbol)
  - modems, portable phones
  - convolutional codes + larger alphabets, such as:
    - 16-QAM
    - 32 Cross
    - 32 Hex
    - 64-QAM

### Coding Theory + Deep-Space Communications

"A Marriage Made in Heaven"

**The value of a dB**

*Technically:*
- extend range
- extend lifetime
- decrease weight
- increase bit rate

*In dollars* (accounting for launch costs, development costs):
- $1,000,000$ in 1970
- $80,000,000$ in 1999