**GEORGIA INSTITUTE OF TECHNOLOGY**

SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

# QUIZ #2

DATE: 13-Mar-13 COURSE: ECE 3084A (Prof. Michaels)

NAME: ______________________________ STUDENT #: ________________

LAST, FIRST

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- Write your name on the front page ONLY. Do not unstaple the test. You should have an additional sheet with Fourier transform tables on one side and Laplace transform tables on the other.
- No calculators, laptops, phones, or other electronic devices allowed. Keep the tables clear of all back backs, books, etc.
- Closed book. However, one page ($8\frac{1}{2}'' \times 11''$) of HAND-WRITTEN notes permitted. OK to write on both sides.
- Unless stated otherwise, justify your reasoning clearly to receive any partial credit.
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.
- The room is small for the number of students in this section. **BE CAREFUL TO NOT LET YOUR EYES WANDER.** Any sort of communication with your fellow students during this exam is strictly forbidden. Any attempt to read off of your neighbor's exam will result in unpleasant disciplinary action.
- Good luck!

| *Problem* | *Value* | *Score* |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |

**Problem Q2.1:**

(5 pts each) The four parts of this problem are unrelated to each other.

(a) Find $X(s)$, the Laplace transform of the signal $x(t) = e^{-2t}[u(t) - u(t-2)]$

(b) Find $x(t)$, the inverse Laplace transform of $X(s) = \dfrac{s+4}{s^2+6s+5}$

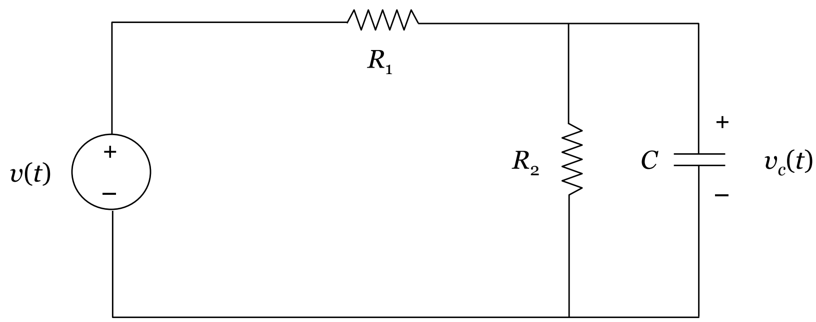
(c) Find $x(t)$, the inverse Laplace transform of $X(s) = \dfrac{3s+20}{s^2+10s+26}$

(d) Use Laplace transforms to solve the following differential equation:

$$\dot{y} + 4y = u(t)$$
$$y(0) = -1$$

**Problem Q2.2:**

(10 pts each) Consider the circuit shown below with input $v(t)$ and output $v_c(t)$:

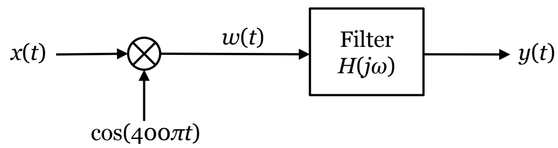


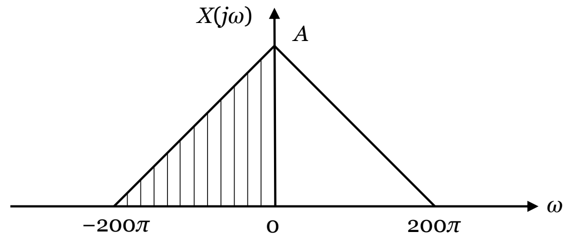

(a) Determine the transfer function of the circuit, $H(s)$, by analyzing it in the $s$-domain. Express $H(s)$ as a ratio of polynomials.

(b) Let $v(t) = V_0 u(t)$. Use the Final Value Theorem to find the steady state voltage across the capacitor.

## Problem Q2.3:

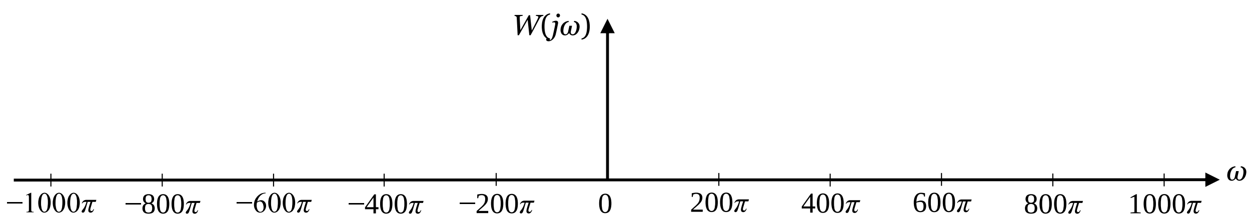
In the modulation/filtering system shown below on the left, assume that the input signal $x(t)$ has a bandlimited Fourier transform as depicted below on the right.




(a) (6 pts) Give the general equation that expresses $W(j\omega)$, the Fourier transform of $w(t)$, in terms of $X(j\omega)$.

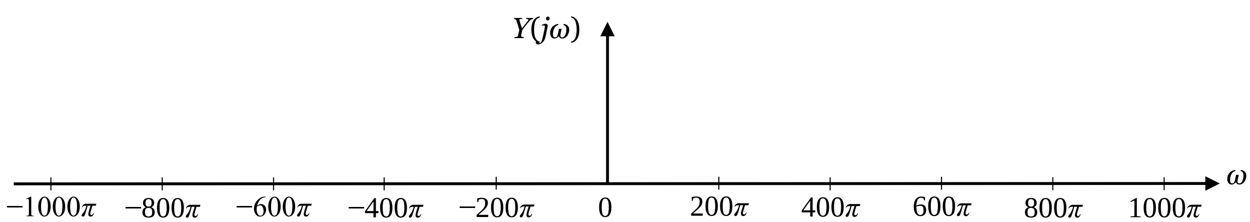
(b) (7 pts) Now carefully plot the the Fourier transform $W(j\omega)$ for the specific $X(j\omega)$ shown above. *Note that the negative frequency portion of $X(j\omega)$ is shaded. Be sure to shade the corresponding regions of $W(j\omega)$, and carefully label both amplitudes and frequencies.*




(c) (7 pts) The frequency response of the filter is

$$H(j\omega) = \begin{cases} 0 & ; \quad |\omega| < 400\pi \\ 1 & ; \quad |\omega| > 400\pi \end{cases}$$

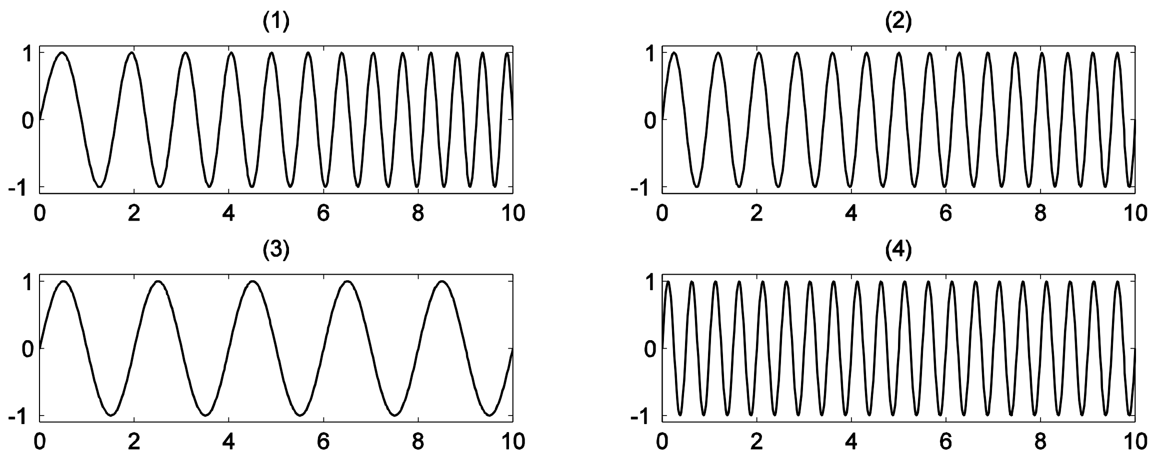
Plot the Fourier transform $Y(j\omega)$ below for the specific $X(j\omega)$ shown above. *Be sure to carefully label both amplitudes and frequencies and be sure to shade the region(s) corresponding to the negative frequencies of the input.* **Also, notice that $H(j\omega)$ is not a lowpass filter.**



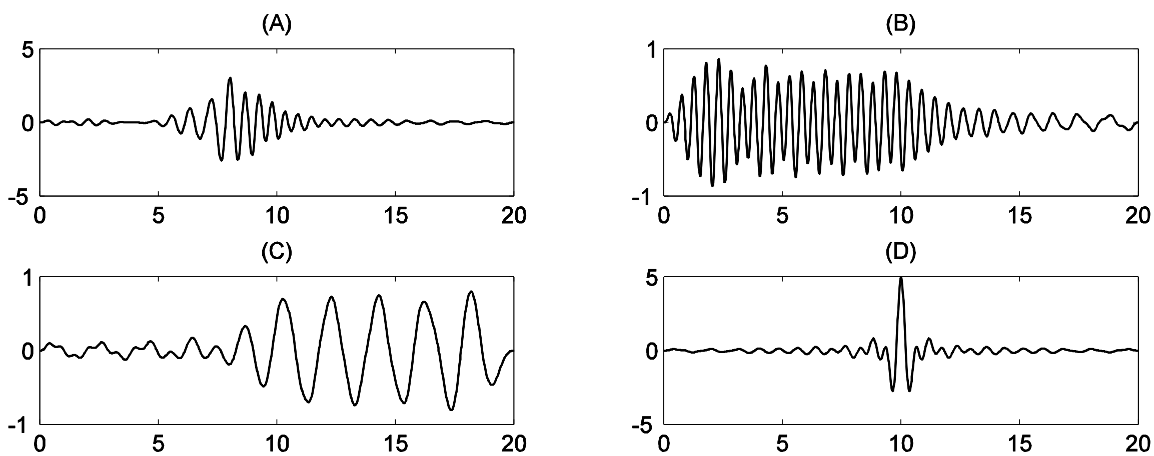
## Problem Q2.4:

(5 pts each) Consider a matched filter where the target signal $x(t)$ is a chirp that starts at 0.5 Hz, ends at 2 Hz, and has a duration of 10 s; $x(t) = \sin(\pi t + 0.15\pi t^2)[u(t) - u(t-10)]$. The impulse response of the matched filter is the time-reversed and time-shifted target signal so that the filter is causal.

The following four signals are inputs to this matched filter. Signal (1) is $x(t)$, signal (2) is a chirp going from 1 Hz to 2 Hz over the 10 s window, signal (3) is a 10 s window of a 0.5 Hz sinusoid, and signal (4) is a 10 s window of a 2 Hz sinusoid.




These four inputs generate the four outputs shown below, although not in the order given.




Indicate the output (A), (B), (C) or (D) corresponding to inputs (1)-(4) in the table below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (1) | | (2) | | (3) | | (4) | |

**Problem Q2.5:**

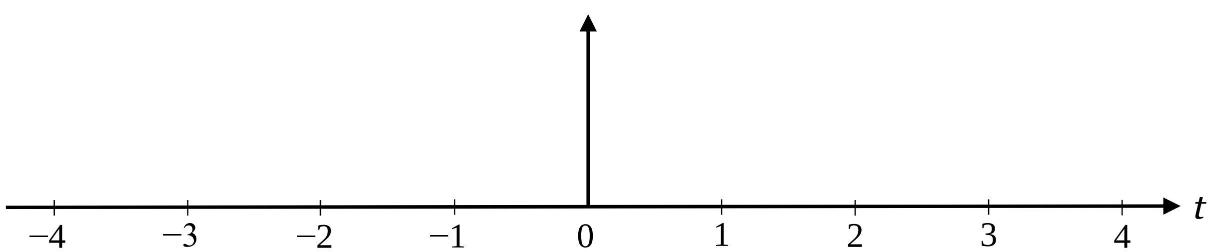
Consider a signal $x(t)$ that is zero outside of the time window $0 < t < T_x$. Its Fourier transform is $X(j\omega)$. Suppose that $X(j\omega)$ is sampled at an increment of $\omega_s$ by multiplying it by an impulse train in the frequency domain:

$$\hat{X}(j\omega) = X(j\omega)P(j\omega) \text{ where } P(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s).$$

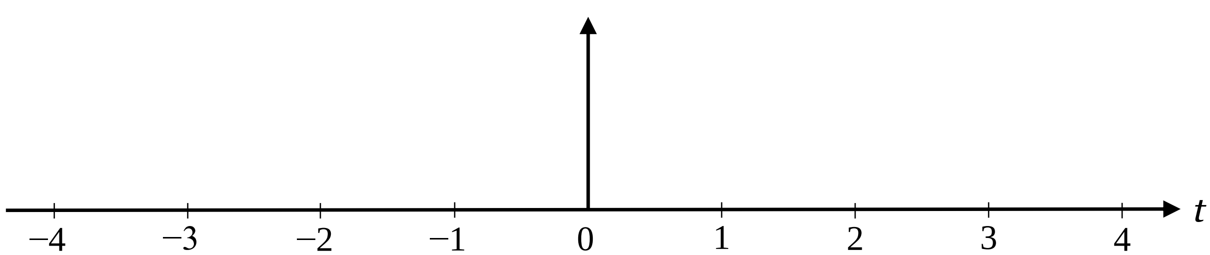
$\hat{X}(j\omega)$ is the sampled spectrum, $\omega_s$ is the frequency sampling interval, and $T_s = 2\pi/\omega_s$.

(a) (10 pts) Show that $\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$, where $\hat{x}(t)$ is the inverse Fourier transform of $\hat{X}(j\omega)$. Show all steps for full credit.

(b) (5 pts) Suppose $x(t) = 1$ for $0 \le t < 1.5$ s ($T_x = 1.5$) and $\omega_s = \pi$. Sketch $\hat{x}(t)$ on the interval $-4 < t < 4$.




(c) (5 pts) For the same $x(t)$ in part (b), sketch $\hat{x}(t)$ on the interval $-4 < t < 4$ for $\omega_s = 2\pi$.



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QUIZ #2

DATE: 13-Mar-13 COURSE: ECE 3084A (Prof. Michaels)

NAME: Answer Key STUDENT #: Section A

LAST, FIRST

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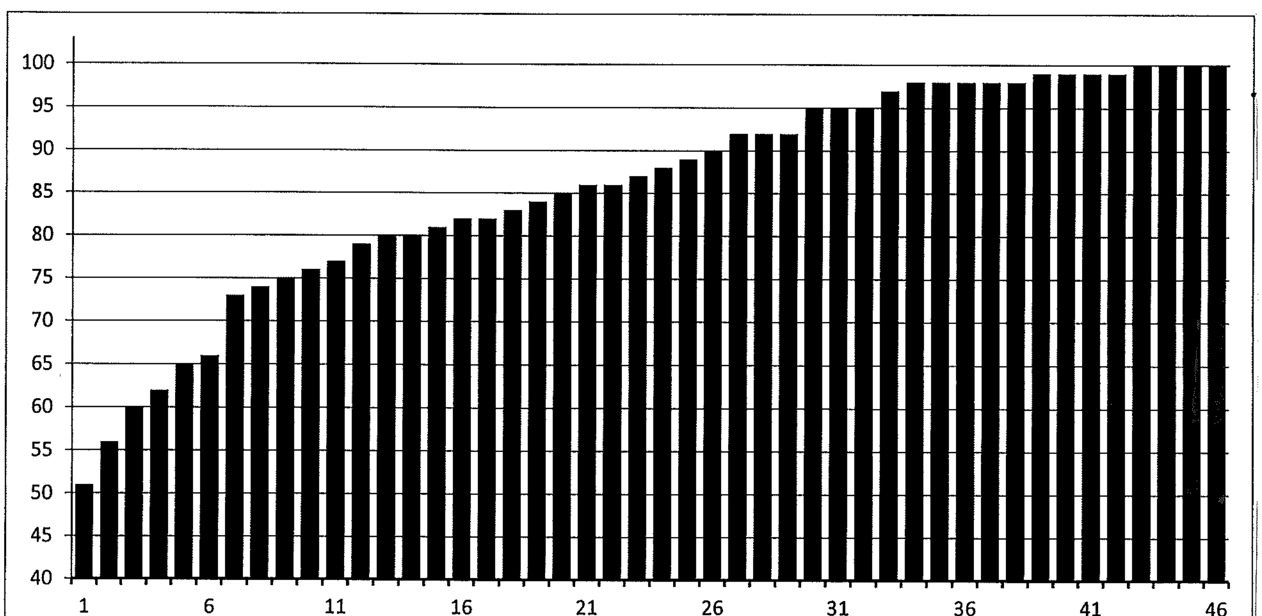


**Problem Q2.1:**

(5 pts each) The four parts of this problem are unrelated to each other.

(a) Find $X(s)$, the Laplace transform of the signal $x(t) = e^{-2t}[u(t) - u(t-2)]$ $= e^{-2t}\tilde{x}(t)$

$X(s) = \tilde{X}(s+2)$

$$\boxed{X(s) = \frac{1 - e^{-2(s+2)}}{s+2}}$$

(b) Find $x(t)$, the inverse Laplace transform of $X(s) = \frac{s+4}{s^2+6s+5}$ $= \frac{s+4}{(s+1)(s+5)}$

$X(s) = \frac{C_1}{s+1} + \frac{C_2}{s+5}$ ; $C_1 = \left.\frac{s+4}{s+5}\right|_{s=-1} = \frac{3}{4}$

$C_2 = \left.\frac{s+4}{s+1}\right|_{s=-5} = \frac{-1}{-4} = \frac{1}{4}$

$$\boxed{x(t) = \frac{3}{4}e^{-t}u(t) + \frac{1}{4}e^{-5t}u(t)}$$

(c) Find $x(t)$, the inverse Laplace transform of $X(s) = \frac{3s+20}{s^s+10s+26}$ $= \frac{3s+20}{(s+5)^2+1}$ $\quad b=5 \quad \omega=1$

$X(s) = \frac{3(s+5)}{(s+5)^2+1} + \frac{5}{(s+5)^2+1}$

$$\boxed{x(t) = 3e^{-5t}\cos(t)u(t) + 5e^{-5t}\sin(t)u(t)}$$

Can also be worked by PFE, but is harder that way

(d) Use Laplace transforms to solve the following differential equation:

$\dot{y} + 4y = u(t)$
$y(0) = -1$

$sY(s) - y(0) + 4Y(s) = \frac{1}{s}$

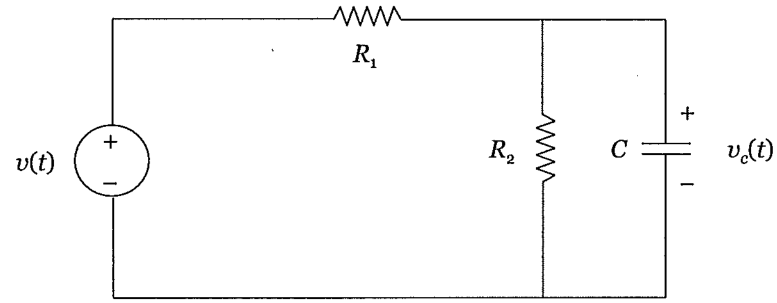
$Y(s)(s+4) = -1 + \frac{1}{s}$

$Y(s) = \frac{-1}{s+4} + \frac{1}{s(s+4)} = \frac{-s+1}{s(s+4)} = \frac{C_1}{s} + \frac{C_2}{s+4}$

$C_1 = \left.\frac{-s+1}{s+4}\right|_{s=0} = \frac{1}{4}$ ; $C_2 = \left.\frac{-s+1}{s}\right|_{s=-4} = -\frac{5}{4}$

$$\boxed{y(t) = \frac{1}{4}u(t) - \frac{5}{4}e^{-4t}u(t)}$$

**Problem Q2.2:**
(10 pts each) Consider the circuit shown below with input $v(t)$ and output $v_c(t)$:




(a) Determine the transfer function of the circuit, $H(s)$, by analyzing it in the $s$-domain.

$$Z_1 = R_1 \qquad Z_2 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{R_2}{R_2Cs+1}$$

Voltage divider

$$V_c(s) = V(s)\,\frac{Z_2}{Z_1+Z_2} = \frac{\left(\frac{R_2}{R_2Cs+1}\right)V(s)}{R_1 + \frac{R_2}{R_2Cs+1}} = \frac{R_2}{R_1R_2Cs + R_1 + R_2}\,V(s)$$

$$\boxed{H(s) = \frac{V_c(s)}{V(s)} = \frac{R_2}{R_1R_2Cs+R_1+R_2}}$$

(b) Let $v(t) = V_0u(t)$. Use the Final Value Theorem to find the steady state voltage across the capacitor.

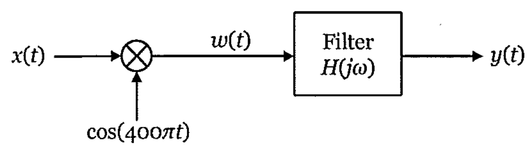
$$\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sV_c(s) = \lim_{s\to 0} s\,H(s)\,\frac{V_o}{s}$$

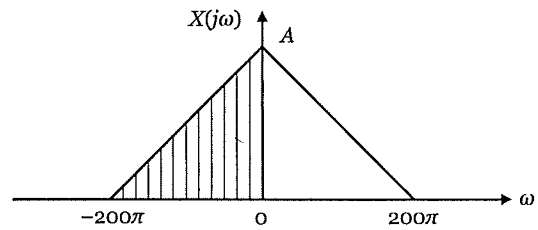
$$= \lim_{s\to 0} V_o H(s) = \boxed{\frac{R_2}{R_1+R_2}V_o}$$

Voltage Divider

## Problem Q2.3:

In the modulation/filtering system shown below on the left, assume that the input signal $x(t)$ has a bandlimited Fourier transform as depicted below on the right.

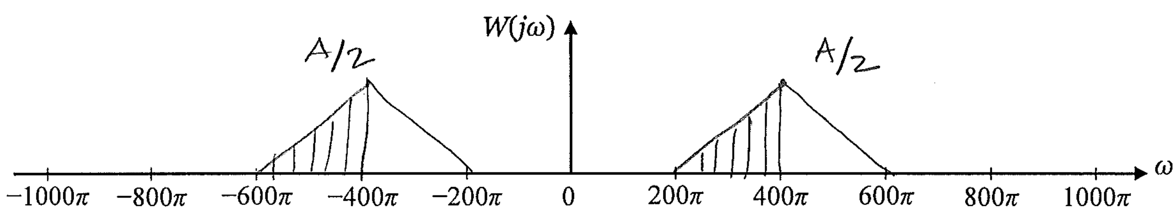



(a) (6 pts) Give the general equation that expresses $W(j\omega)$, the Fourier transform of $w(t)$, in terms of $X(j\omega)$.

$$W(j\omega) = \frac{1}{2\pi} X(j\omega) * [\pi\delta(\omega+400\pi) + \pi\delta(\omega-400\pi)]$$
$$= \boxed{\frac{1}{2}X(j(\omega+400\pi)) + \frac{1}{2}X(j(\omega-400\pi))}$$

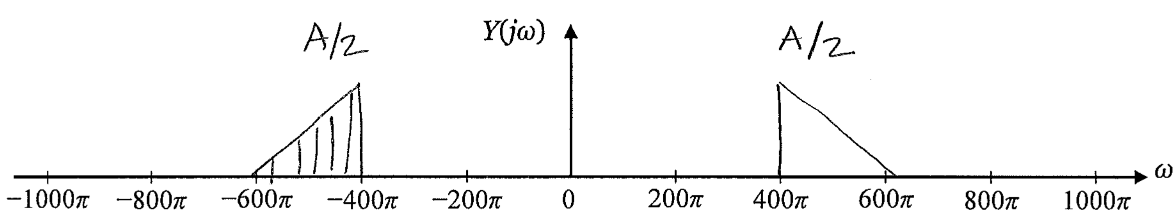
(b) (7 pts) Now carefully plot the the Fourier transform $W(j\omega)$ for the specific $X(j\omega)$ shown above. *Note that the negative frequency portion of $X(j\omega)$ is shaded. Be sure to shade the corresponding regions of $W(j\omega)$, and carefully label both amplitudes and frequencies.*




(c) (7 pts) The frequency response of the filter is

$$H(j\omega) = \begin{cases} 0 & ; \quad |\omega| < 400\pi \\ 1 & ; \quad |\omega| > 400\pi \end{cases}$$

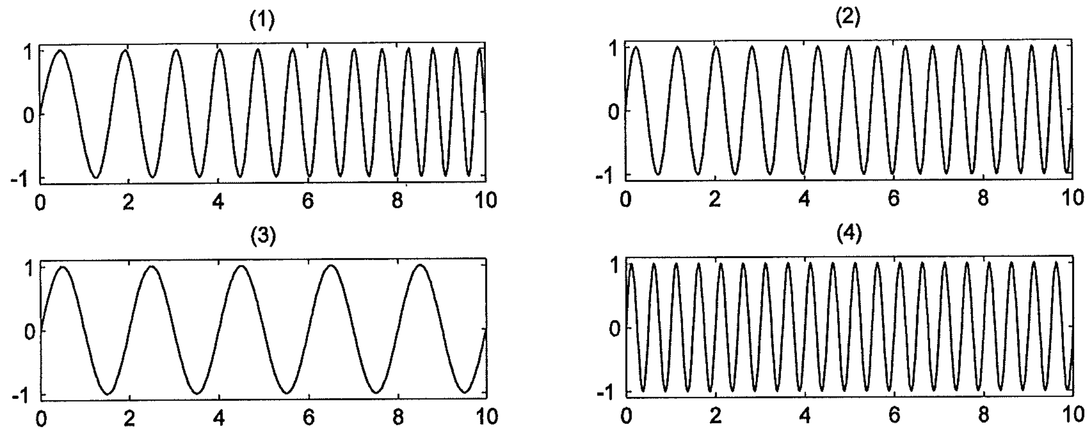
Plot the Fourier transform $Y(j\omega)$ below for the specific $X(j\omega)$ shown above. *Be sure to carefully label both amplitudes and frequencies and be sure to shade the region(s) corresponding to the negative frequencies of the input.* **Also, notice that $H(j\omega)$ is not a lowpass filter.**



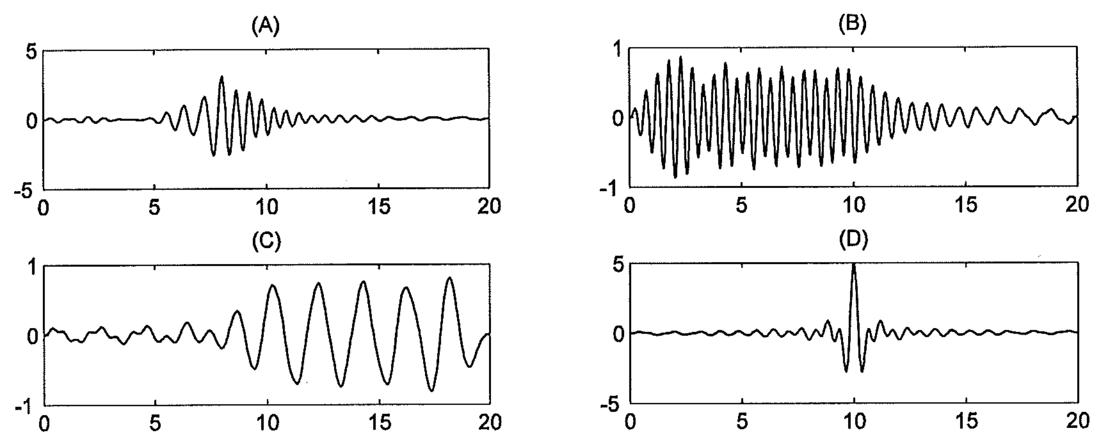
## Problem Q2.4:

(5 pts each) Consider a matched filter where the target signal $x(t)$ is a chirp that starts at 0.5 Hz, ends at 2 Hz, and has a duration of 10 s; $x(t) = \sin(\pi t + 0.15\pi t^2)[u(t) - u(t-10)]$. The impulse response of the matched filter is the time-reversed and time-shifted target signal so that the filter is causal.

The following four signals are inputs to this matched filter. Signal (1) is $x(t)$, signal (2) is a chirp going from 1 Hz to 2 Hz over the 10 s window, signal (3) is a 10 s window of a 0.5 Hz sinusoid, and signal (4) is a 10 s window of a 2 Hz sinusoid.




These four inputs generate the four outputs shown below, although not in the order given.




Indicate the output (A), (B), (C) or (D) corresponding to inputs (1)-(4) in the table below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (1) | D | (2) | A | (3) | C | (4) | B |

## Problem Q2.5:

Consider a signal $x(t)$ that is zero outside of the time window $0 < t < T_x$. Its Fourier transform is $X(j\omega)$. Suppose that $X(j\omega)$ is sampled at an increment of $\omega_s$ by multiplying it by an impulse train in the frequency domain:

$$\hat{X}(j\omega) = X(j\omega)P(j\omega) \text{ where } P(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s).$$

$\hat{X}(j\omega)$ is the sampled spectrum, $\omega_s$ and $T_s = 2\pi/\omega_s$.

(a) (10 pts) Show that $\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$, where $\hat{x}(t)$ is the inverse Fourier transform of $\hat{X}(j\omega)$. Show all steps for full credit.

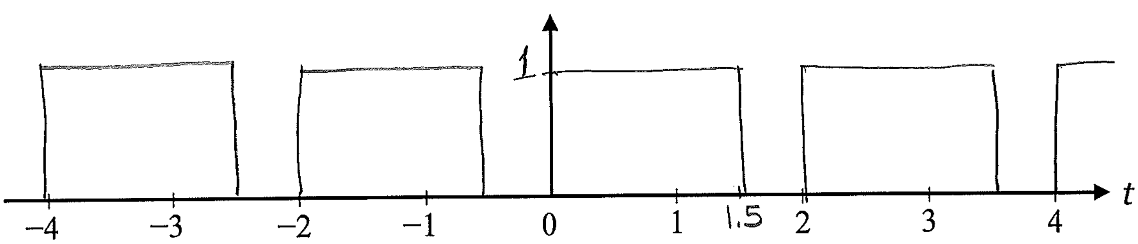
$$\hat{x}(t) = x(t) * p(t) \qquad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{From F.T. Table}$$

$$= x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(t - nT_s)$$

(b) (5 pts) Suppose $x(t) = 1$ for $0 \le t < 1.5$ s ($T_x = 1.5$) and $\omega_s = \pi$. Sketch $\hat{x}(t)$ on the interval $-4 < t < 4$.

$$T_s = \frac{2\pi}{\pi} = 2\ s$$




(c) (5 pts) For the same $x(t)$ in part (b), sketch $\hat{x}(t)$ on the interval $-4 < t < 4$ for $\omega_s = 2\pi$.

$$T_s = \frac{2\pi}{2\pi} = 1\ s$$