GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL & COMPUTER ENGINEERING

QUIZ #2

DATE: 06-Nov-13 COURSE: ECE 3084A (Prof. Michaels)

NAME: ______________________________ STUDENT #: ______________

LAST, FIRST

- Write your name on the front page ONLY. Do not unstaple the test. You should have an additional sheet with Fourier transform tables on one side and Laplace transform tables on the other.
- No calculators, laptops, phones, or other electronic devices allowed. Keep the desks clear of all backpacks, books, etc.
- This is a *closed book* exam. However, one page ($8\frac{1}{2}'' \times 11''$) of HAND-WRITTEN notes is permitted; it is OK to write on both sides.
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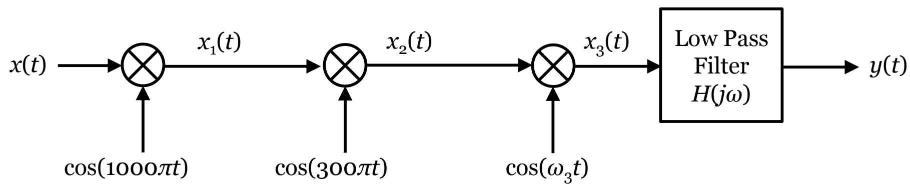
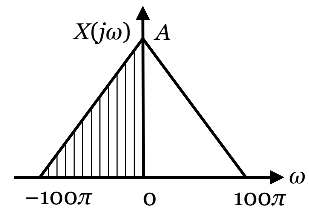
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.
- The room is small for the number of students in this section. **BE CAREFUL TO NOT LET YOUR EYES WANDER.** Any sort of communication with your fellow students during this exam is strictly forbidden. Any attempt to read off of your neighbor's exam will result in unpleasant disciplinary action.
- Good luck!

Problem	*Value*	*Score*
1	20	
2	20	
3	20	
4	20	
5	20	

Problem Q2.1:
Consider the system shown below where the input $x(t)$ is modulated three times and then low-pass filtered. Assume that $x(t)$ has the bandlimited spectrum shown on the right.





(a) (10 pts) Sketch $X_1(j\omega)$ and $X_2(j\omega)$, the Fourier transforms of $x_1(t)$ and $x_2(t)$ after the first two modulators.

(b) (5 pts) If the goal is for the output $y(t)$ to equal the input $x(t)$, what is the lowest possible value of ω_3 to achieve this? (Note that ω_3 is the frequency of the third modulator.)

(c) (5 pts) Sketch $H(j\omega)$, the frequency response of a low pass filter, such that $y(t) = x(t)$ for the value of ω_3 you found in part (b).

Problem Q2.2:

Let $x(t)$ be a signal that is sampled by multiplication with an impulse train; i.e., $x_s(t) = x(t)p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$.

(a) (5 pts) Sketch $X(j\omega)$ for $x(t) = \dfrac{\sin(t)}{\pi t}$ over the range of $-8 < \omega < 8$.

(b) (5 pts) Sketch $X_s(j\omega)$ for the $x(t)$ in part (a) and for $T_s = \frac{2\pi}{3}$ ($\omega_s = 3$) over the range of $-8 < \omega < 8$.

(c) (5 pts) If $x_s(t)$ is passed through an ideal low-pass with a magnitude of T_s and a cutoff frequency of $\omega_c = 1.5$ r/s, what is $x_r(t)$, the output of the filter? Does $x_r(t) = x(t)$?

(d) (5 pts) Now suppose that $x(t) = e^{-t}u(t)$. Will the filter ouput $x_r(t) = x(t)$ for the same T_s and low-pass filter of parts (b) and (c)? Explain your reasoning for full credit.

Problem Q2.3:

(5 pts each) The four parts of this problem are unrelated to each other.

(a) Find $X(s)$, the Laplace transform of $x(t) = t^2 u(t-1)$.

(b) Find $x(t)$, the inverse Laplace transform of $X(s) = \dfrac{e^{-2s}}{(s+2)^2}$.

(c) Find $x(t)$, the inverse Laplace transform of $X(s) = \dfrac{s-1}{s+1}$.

(d) Find $x(t)$, the inverse Laplace transform of $X(s) = \dfrac{4}{s^2(s+2)}$.

Problem Q2.4:

(5 pts each) The functions numbered (1) through (11) in the table below are possible terms that could be present in the outputs of the four systems shown below. The transfer function of each system is given by $H(s)$ and the input is $x(t)$. In the space below each system, indicate the number(s) of the functions that are present in the output $y(t)$, where $Y(s) = H(s)X(s)$. Each system will generally have multiple terms. Select "(12) other" if you believe a term is present in the output that is not listed in the table.

(1) $u(t)$	(2) $tu(t)$	(3) $t^2u(t)$	(4) $e^{-t}u(t)$
(5) $te^{-t}u(t)$	(6) $t^2e^{-t}u(t)$	(7) $e^{t}u(t)$	(8) $te^{t}u(t)$
(9) $\cos(t+\theta)u(t)$	(10) $t\cos(t+\phi)u(t)$	(11) $t^2\cos(t+\psi)u(t)$	(12) other

Example: $H(s) = \dfrac{1}{s+1}$ and $x(t) = u(t)$. $y(t) = c_1u(t) + c_2e^{-t}u(t)$. Answer: (1) and (4)

(a) $H(s) = \dfrac{1}{s}$ and $x(t) = te^{-t}u(t)$.

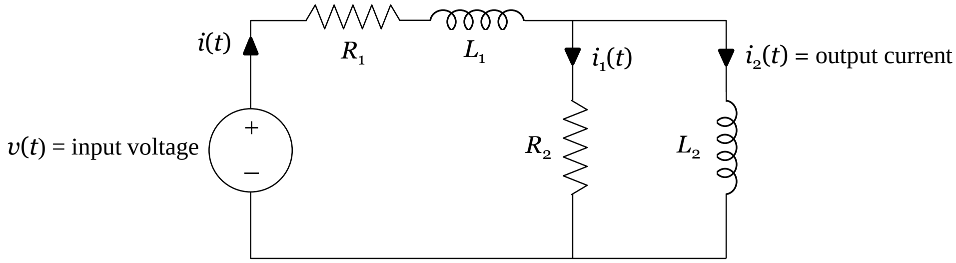
(b) $H(s) = \dfrac{1}{s^2+1}$ and $x(t) = u(t)$.

(c) $H(s) = \dfrac{1}{s(s^2+1)}$ and $x(t) = \cos(t)u(t)$.

(d) $H(s) = \dfrac{1}{(s^2-1)}$ and $x(t) = e^{-t}u(t)$.

Problem Q2.5:

Consider the circuit shown below where the input is the voltage $v(t)$ and the output is the current through the the second inductor, $i_2(t)$.



(a) (5 pts) Draw this circuit in the s-domain assuming zero initial conditions.

(b) (10 pts) What is $H(s)$, the transfer function of this circuit relating the input $v(t)$ to the output $i_2(t)$? Express it as a ratio of polynomials in s.

(c) (5 pts) Assume that $v(t) = V_0 u(t)$ and that there are zero initial conditions. Either use the final value theorem to find $\lim_{t \to \infty} i_2(t)$, or explain why the limit doesn't exist.

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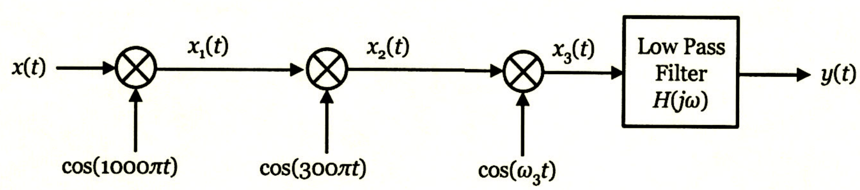
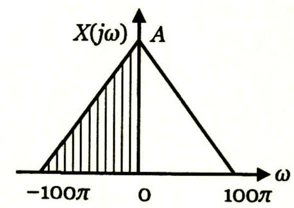
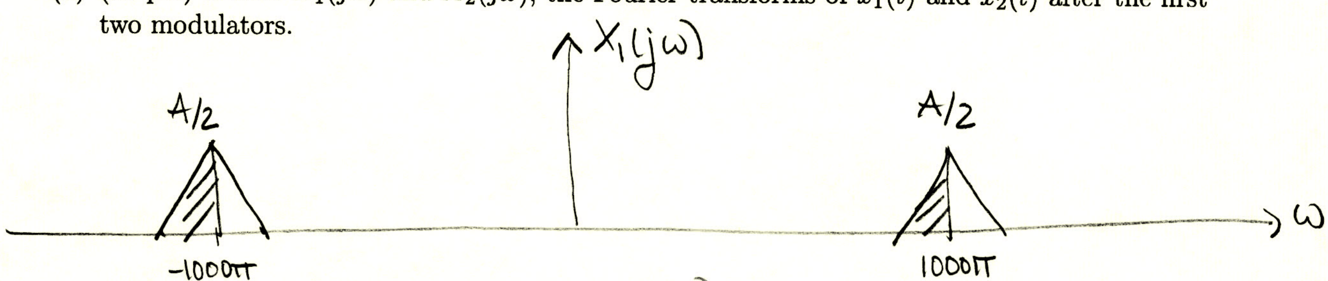
Problem Q2.1:

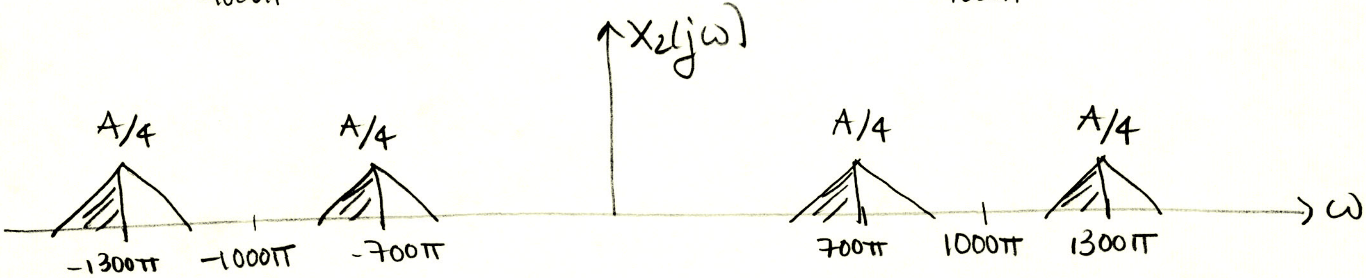
Consider the system shown below where the input $x(t)$ is modulated three times and then low-pass filtered. Assume that $x(t)$ has the bandlimited spectrum shown on the right.





(a) (10 pts) Sketch $X_1(j\omega)$ and $X_2(j\omega)$, the Fourier transforms of $x_1(t)$ and $x_2(t)$ after the first two modulators.

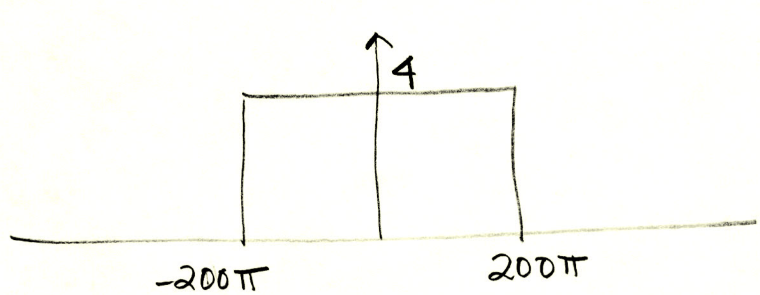




(b) (5 pts) If the goal is for the output $y(t)$ to equal the input $x(t)$, what is the lowest possible value of ω_3 to achieve this? (Note that ω_3 is the frequency of the third modulator.)

$\omega_3 = 700\pi$ will move the replicas at $\pm 700\pi$ back to zero. The replicas at $\pm 1300\pi$ will not interfere.

(c) (5 pts) Sketch $H(j\omega)$, the frequency response of a low pass filter, such that $y(t) = x(t)$ for the value of ω_3 you found in part (b).



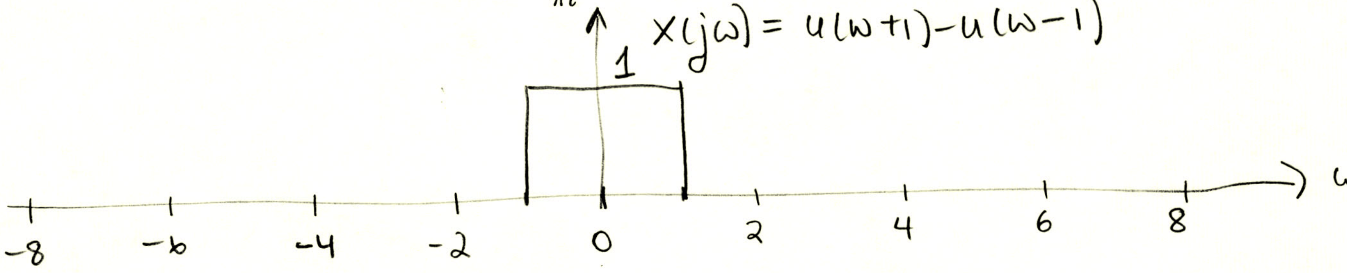
Cutoff can be from 100π to 600π.

Problem Q2.2:

Let $x(t)$ be a signal that is sampled by multiplication with an impulse train; i.e., $x_s(t) = x(t)p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$.

(a) (5 pts) Sketch $X(j\omega)$ for $x(t) = \frac{\sin(t)}{\pi t}$ over the range of $-8 < \omega < 8$.

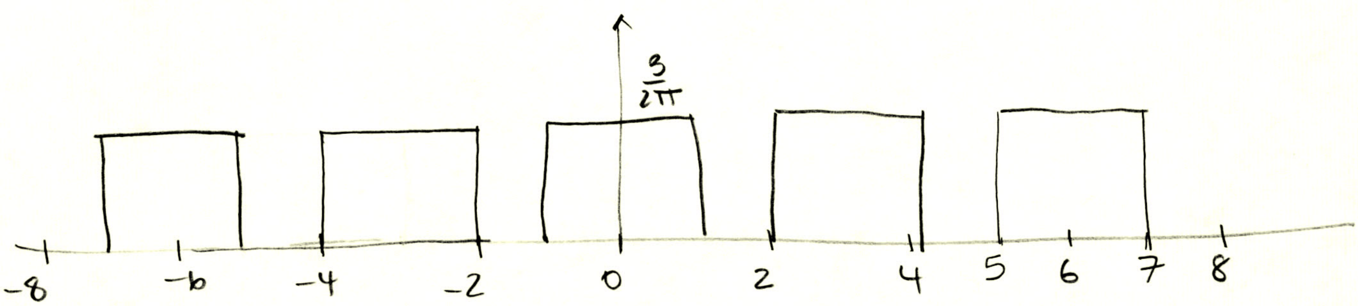
$X(j\omega) = u(\omega+1) - u(\omega-1)$



(b) (5 pts) Sketch $X_s(j\omega)$ for the $x(t)$ in part (a) and for $T_s = \frac{2\pi}{3}$ ($\omega_s = 3$) over the range of $-8 < \omega < 8$.

$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T_s} k)$ $\qquad \frac{2\pi}{T_s} = \omega_s = 3$

$X_s(j\omega) = \frac{3}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - 3k)$



(c) (5 pts) If $x_s(t)$ is passed through an ideal low-pass with a magnitude of T_s and a cutoff frequency of $\omega_c = 1.5$ r/s, what is $x_r(t)$, the output of the filter? Does $x_r(t) = x(t)$?

$T_s = \frac{2\pi}{3} \qquad X_r(j\omega) = X(j\omega)$

$x_r(t) = \frac{\sin(t)}{\pi t} = x(t)$

(d) (5 pts) Now suppose that $x(t) = e^{-t}u(t)$. Will the filter ouput $x_r(t) = x(t)$ for the same T_s and low-pass filter of parts (b) and (c)? Explain your reasoning for full credit.

No because this $x(t)$ is not bandlimited. The low pass filter will include pieces of the replicas of $X(j\omega)$ that overlap will the original.

Problem Q2.3:
(5 pts each) The four parts of this problem are unrelated to each other.

(a) Find $X(s)$, the Laplace transform of $x(t) = t^2u(t-1)$.

$$x(t) = (t^2-2t+1)u(t-1) + 2t\,u(t-1) - u(t-1)$$
$$= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1)$$
$$X(s) = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]$$
$$= e^{-s}\left[\frac{s^2+2s+2}{s^3}\right]$$

(b) Find $x(t)$, the inverse Laplace transform of $X(s) = \frac{e^{-2s}}{(s+2)^2}$.

$$\frac{1}{(s+2)^2} \longleftrightarrow t e^{-2t}u(t)$$
$$x(t) = (t-2)e^{-2(t-2)}u(t-2)$$

(c) Find $x(t)$, the inverse Laplace transform of $X(s) = \frac{s-1}{s+1}$.

$$\begin{array}{r|l} & 1 \\ \hline s+1 & s-1 \\ & s+1 \\ \hline & -2 \end{array}$$

$$X(s) = 1 - \frac{2}{s+1}$$
$$x(t) = \delta(t) - 2e^{-t}u(t)$$

(d) Find $x(t)$, the inverse Laplace transform of $X(s) = \frac{4}{s^2(s+2)}$.

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s+2}$$
$$C_2 = \left.\frac{4}{s+2}\right|_{s=0} = 2\ ; \quad C_3 = \left.\frac{4}{s^2}\right|_{s=-2} = 1$$
$$4 = C_1 s(s+2) + C_2(s+2) + C_3(s^2)$$
equate coefficients of s^2: $0 = C_1 + C_3\ ;\ C_1 = -1$
$$x(t) = -u(t) + 2t\,u(t) + e^{-2t}u(t)$$

Problem Q2.4:

(5 pts each) The functions numbered (1) through (11) in the table below are possible terms that could be present in the outputs of the four systems shown below. The transfer function of each system is given by $H(s)$ and the input is $x(t)$. In the space below each system, indicate the number(s) of the functions that are present in the output $y(t)$, where $Y(s) = H(s)X(s)$. Each system will generally have multiple terms. Select "(12) other" if you believe a term is present in the output that is not listed in the table.

(1) $u(t)$	(2) $tu(t)$	(3) $t^2u(t)$	(4) $e^{-t}u(t)$
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(9) $\cos(t+\theta)u(t)$	(10) $t\cos(t+\phi)u(t)$	(11) $t^2\cos(t+\psi)u(t)$	(12) other

Example: $H(s) = \dfrac{1}{s+1}$ and $x(t) = u(t)$. $y(t) = c_1u(t) + c_2e^{-t}u(t)$. Answer: (1) and (4)

(a) $H(s) = \dfrac{1}{s}$ and $x(t) = te^{-t}u(t)$.

$$Y(s) = \frac{1}{s}\cdot\frac{1}{(s+1)^2} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{(s+1)^2}$$

(1) (4) (5)

(b) $H(s) = \dfrac{1}{s^2+1}$ and $x(t) = u(t)$.

$$Y(s) = \frac{1}{s^2+1}\cdot\frac{1}{s} = \frac{C_1}{s} + \frac{C_2 + C_3 s}{s^2+1}$$

(1) (9)

(c) $H(s) = \dfrac{1}{s(s^2+1)}$ and $x(t) = \cos(t)u(t)$.

$$Y(s) = \frac{1}{s(s^2+1)}\cdot\frac{s}{s^2+1} = \frac{1}{(s^2+1)^2}$$

(9) (10)

Could also get u(t) term if there are initial conditions

Full credit for (9) & (10) or (1), (9) & (10)

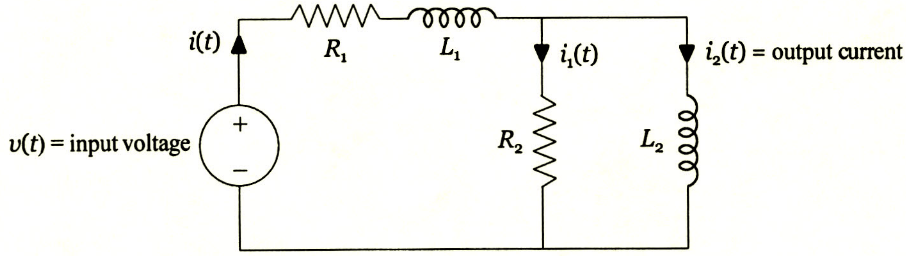
(d) $H(s) = \dfrac{1}{(s^2-1)}$ and $x(t) = e^{-t}u(t)$.

$$Y(s) = \frac{1}{s^2-1}\cdot\frac{1}{s+1} = \frac{1}{(s+1)^2(s-1)} = \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{C_3}{s-1}$$

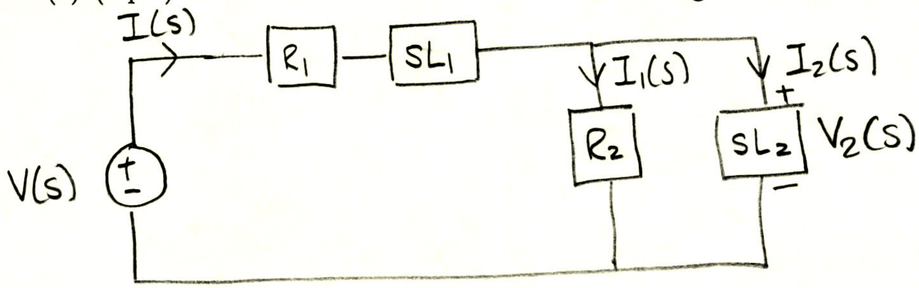
(4) (5) (7)

Problem Q2.5:

Consider the circuit shown below where the input is the voltage $v(t)$ and the output is the current through the the second inductor, $i_2(t)$.



(a) (5 pts) Draw this circuit in the s-domain assuming zero initial conditions.



(b) (10 pts) What is $H(s)$, the transfer function of this circuit relating the input $v(t)$ to the output $i_2(t)$? Express it as a ratio of polynomials in s.

Voltage divider: $V_2(s) = V(s)\dfrac{R_2 \| sL_2}{R_1 + sL_1 + R_2 \| sL_2} = \dfrac{R_2 s L_2 \, V(s)}{(R_2 + sL_2)\left[R_1 + sL_1 + \dfrac{R_2 s L_2}{R_2 + sL_2}\right]}$

$$V_2(s) = \frac{R_2 s L_2 \, V(s)}{(R_2 + sL_2)(R_1 + sL_1) + R_2 s L_2}$$

$$I_2(s) = \frac{V_2(s)}{sL_2} = \frac{R_2 \, V(s)}{L_1 L_2 s^2 + (R_1 L_2 + R_2 L_1 + R_2 L_2)s + R_1 R_2}$$

$$H(s) = \frac{I_2(s)}{V(s)} = \frac{R_2}{L_1 L_2 s^2 + (R_1 L_2 + R_2 L_1 + R_2 L_2)s + R_1 R_2}$$

Many other ways to get $H(s)$

(c) (5 pts) Assume that $v(t) = V_0 u(t)$ and that there are zero initial conditions. Either use the final value theorem to find $\lim_{t\to\infty} i_2(t)$, or explain why the limit doesn't exist.

$V(s) = \dfrac{V_0}{s}$; $I_2(s) = H(s)\dfrac{V_0}{s}$

F.V.T. $\lim_{t\to\infty} i_2(t) = \lim_{s\to 0} s I_2(s) = V_0 H(s=0) = \dfrac{V_0 R_2}{R_1 R_2} = \dfrac{V_0}{R_1}$

easy to check since inductors → shorts as $t \to \infty$