GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL & COMPUTER ENGINEERING
FINAL EXAM

DATE: 30-Apr-14 COURSE: ECE 3084A (Prof. Michaels)

NAME: ______________________________ STUDENT #: ______________
LAST, FIRST

- Write your name on the front page ONLY. Do not unstaple the test. You should have an additional sheet with Fourier transform tables on one side and Laplace transform tables on the other.
- You may use a calculator, but no laptops, phones, or other electronic devices are allowed. Keep the tables clear of all back backs, books, etc.
- This is a closed book exam. However, two pages ($8\frac{1}{2}'' \times 11''$) of HAND-WRITTEN notes are permitted. It's OK to write on both sides.
- Unless stated otherwise, justify your reasoning clearly to receive any partial credit.
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.
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Please sign the following statement: **One or more students will be taking the final at a later time, and I will not discuss its contents with anyone until the solutions are posted. I understand that if I do, it is a violation of the student honor code.**

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Problem	*Value*	*Score*
1	30	
2	40	
3	30	
4	20	
5	30	
6	25	
7	25	
Total	200	

Problem F.1:

(5 pts each) Short problem assortment. The six parts of this problem are unrelated to each other.

(a) Consider the system with input $x(t)$ and output $y(t)$ described by the equation $y(t) = x(2t)$. Circle "yes" or "no" in each row of the table below to indicate whether or not this system is linear, time-invariant, and/or causal.

Linear	yes	no
Time-Invariant	yes	no
Causal	yes	no

(b) Consider the signal $x(t) = t[u(t) - u(t-1)]$. Sketch $x(t)$, $x(-t+1)$, and $x(2t-5)$ in the space provided below.

(c) Simplify the following expression as much as possible:

$$\frac{d}{dt}\cos(\pi t)[u(t) - u(t-2)] =$$

(d) Simplify the following expression as much as possible:

$$\int_{-\infty}^{t} \tau^3[\delta(\tau+1)+\delta(\tau-2)]d\tau =$$

(e) Find $X(j\omega)$, the Fourier transform of $x(t) = e^{-|t|}$.

(f) Find and sketch $x(t)$, the inverse Fourier transform of $X(j\omega) = \dfrac{4\sin^2(\omega/2)}{\omega^2}$.

Problem F.2:

(10 points each) Consider the convolution $y(t) = x(t) * h(t)$. For each part, $x(t)$ and $h(t)$ are given. Sketch $x(t)$, $h(t)$, and $y(t)$ in the space provided with correct labels for all axes. You do not have to provide equations for your answers since you may be able to do some or all of these by inspection.

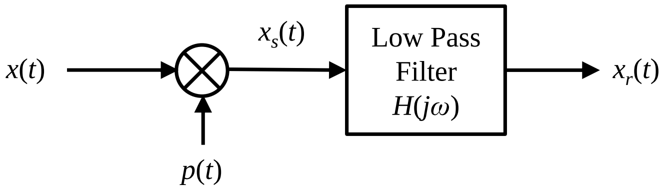
(a) $x(t) = u(t)$ and $h(t) = u(t-2)$

(b) $x(t) = 3\delta(t) + 2\delta(t-1) + \delta(t-2)$ and $h(t) = \delta(t) - \delta(t-5)$

(c) $x(t) = u(t-1) - u(t-2)$ and $h(t) = 2[u(t) - u(t-4)]$

(d) $x(t) = u(t) - u(t-1)$ and $h(t) = \sin(2\pi t)u(t)$

Problem F.3:
This problem considers sampling and reconstruction as shown in the figure below where $x(t)$ is the signal being sampled, $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ is an impulse train, $H(j\omega)$ is an ideal low pass filter with cutoff frequency $\omega_s/2$ and gain T_s, and $x_r(t)$ is the reconstructed signal. For all parts of this problem the input is the periodic signal $x(t) = \cos(\omega_0 t) + 0.3\cos(3\omega_0 t)$, and the fundamental frequency $f_0 = \omega_0/(2\pi)$ is 196 Hz. This signal is a (poorly) synthesized version of a fictitious musical instrument playing the note "G" below middle "C".



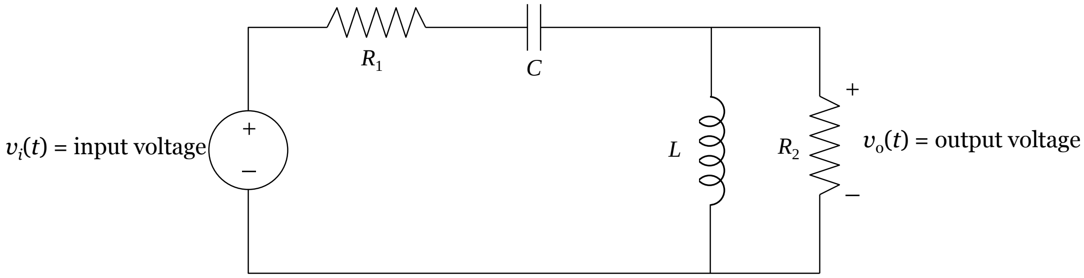
(a) (10 pts) Find and accurately sketch $X(j\omega)$, the Fourier transform of $x(t)$, as a function of $f = \omega/2\pi$.

(b) (5 pts) For what choices of sampling frequency f_s will $x_r(t) = x(t)$?

(c) (10 pts) Now let the sampling frequency be $f_s = 1000$ Hz. Accurately sketch $X_s(j\omega)$, the Fourier transform of the sampled signal $x_s(t)$, as a function of f for $-1000 \leq f \leq 1000$ Hz. Find an equation for the reconstructed signal $x_r(t)$ for this value of f_s.

(d) (5 pts) Is the reconstructed signal $x_r(t)$ periodic for $f_s = 1000$ Hz? If so, what is its fundamental frequency? If not, explain why not.

Problem F.4:
Consider the circuit shown below with input voltage source $v_i(t)$ and output voltage $v_o(t)$:



(a) (5 pts) Draw this circuit in the s-domain assuming zero initial conditions. Properly label all quantities indicated on the original circuit for full credit.

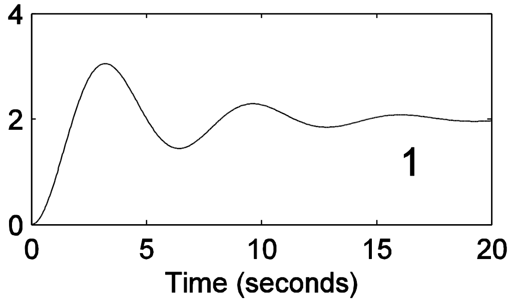
(b) (10 pts) Determine the transfer function relating the input voltage source to the voltage across R_2, $H(s) = V_o(s)/V_i(s)$, by analyzing the circuit in the s-domain. Simplify $H(s)$ so that it is expressed as a ratio of polynomials in s with all like terms combined.

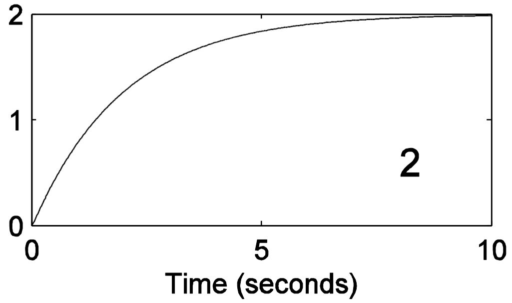
(c) (5 pts) Let $v_i(t) = V_0 u(t)$. Use the Final Value Theorem to find the steady state output voltage. Make sure that this value makes sense!

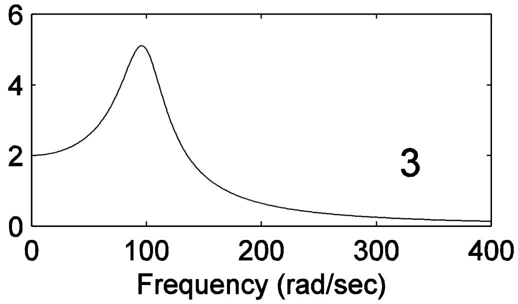
Problem F.5:

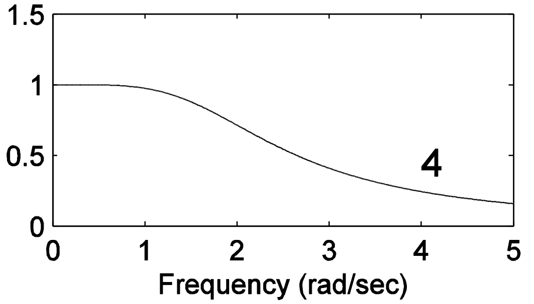
(5 pts each) Consider the six low pass filters whose transfer functions are given below. Your job is to select either a step response (graphs 1 and 2) or a frequency response (graphs 3 through 6) from the numbered choices below that matches the transfer function. Each one will have exactly one match (that is, there will be either a step response or a frequency response that matches, but not both). Put the number of the matching graph in the space provided.

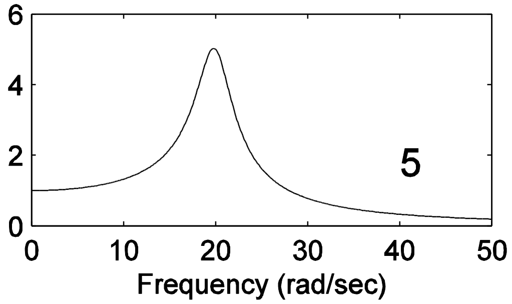
Transfer Function	Graph	Transfer Function	Graph
(a) $H(s) = \dfrac{20000}{s^2 + 40s + 10000}$		(d) $H(s) = \dfrac{2}{s^2 + 0.4s + 1}$	
(b) $H(s) = \dfrac{50}{s^2 + 50s + 25}$		(e) $H(s) = \dfrac{4}{s^2 + 2.8s + 4}$	
(c) $H(s) = \dfrac{2}{s^2 + 2s + 1}$		(f) $H(s) = \dfrac{400}{s^2 + 4s + 400}$	

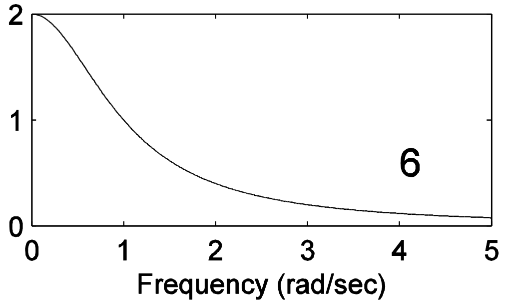






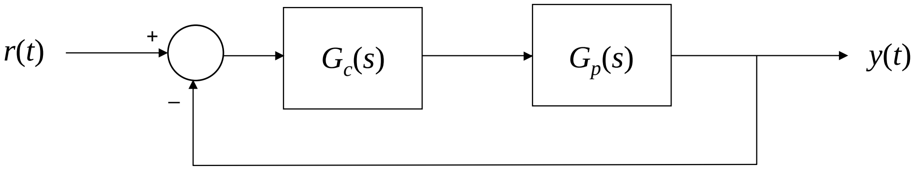






Problem F.6:

This problem considers control of a plant $G_p(s) = \dfrac{s-b}{s^2+6s+8}$ using a proportional controller $G_c(s) = K_p$.



(a) (10 pts) Find the transfer function $H(s)$ of the closed loop system in terms of b, the plant zero location, and K_p, the proportional gain. Express your answer as a ratio of polynomials in s with all like terms combined.

(b) (5 pts) For $b = 2$, find all $K_p > 0$ for which the closed loop system is stable.

(c) (10 pts) Find the complete step response, $y(t)$ for $r(t) = u(t)$, for $b = 2$ and $K_p = 1$. Comment on the ability of the closed loop system to track a step input.

Problem F.7:

Consider the same feedback system of the previous problem with $G_p(s) = \dfrac{s+2}{s^2+9s+14}$ and a PI controller, $G_c(s) = K_p + K_i/s$.

(a) (10 pts) Find the transfer function $H(s)$ of the closed loop system in terms of the controller gains, K_p and K_i. Express your answer as a ratio of polynomials in s with all like terms combined.

(b) (5 pts) What are the restrictions, if any, on K_p and K_i for the system to achieve perfect tracking of a step input in the steady-state?

(c) (10 pts) Suppose you want the closed loop system to have one pole at $s = -10$ and two repeated poles at $s = -2$. Either find $K_p > 0$ and $K_i > 0$ that will achieve this goal, or show that it can't be done.

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Problem F.1:

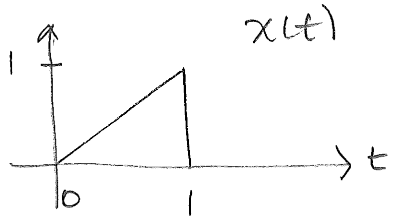
(5 pts each) Short problem assortment. The six parts of this problem are unrelated to each other.

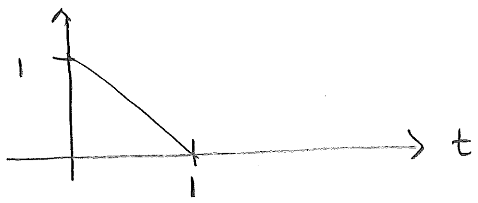
(a) Consider the system with input $x(t)$ and output $y(t)$ described by the equation $y(t) = x(2t)$. Circle "yes" or "no" in each row of the table below to indicate whether or not this system is linear, time-invariant, and/or causal.

Linear	[✓ yes]	no
Time-Invariant	yes	[✓ no]
Causal	yes	[✓ no]

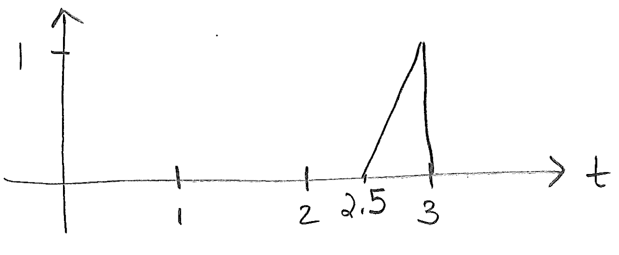
(b) Consider the signal $x(t) = t[u(t) - u(t-1)]$. Sketch $x(t)$, $x(-t+1)$, and $x(2t-5)$ in the space provided below.

$x(-t+1) = x(-(t-1))$ flipped, shifted





$x(2t-5) = x(2(t-\frac{5}{2}))$ scaled & shifted



(c) Simplify the following expression as much as possible:

$$\frac{d}{dt}\left\{\cos(\pi t)[u(t) - u(t-2)]\right\} = \cos(\pi t)[\delta(t) - \delta(t-2)] - \pi\sin(\pi t)[u(t) - u(t-2)]$$

$$= \delta(t) - \delta(t-2) - \pi\sin(\pi t)[u(t) - u(t-2)]$$

(d) Simplify the following expression as much as possible:

$$\int_{-\infty}^{t} \tau^3[\delta(\tau+1)+\delta(\tau-2)]d\tau = \int_{-\infty}^{t}[-\delta(\tau+1)+8\delta(\tau-2)]d\tau$$

$$= -u(t+1)+8u(t+2)$$

(e) Find $X(j\omega)$, the Fourier transform of $x(t) = e^{-|t|}$.

$$x(t) = e^{-t}u(t) + e^{t}u(-t)$$

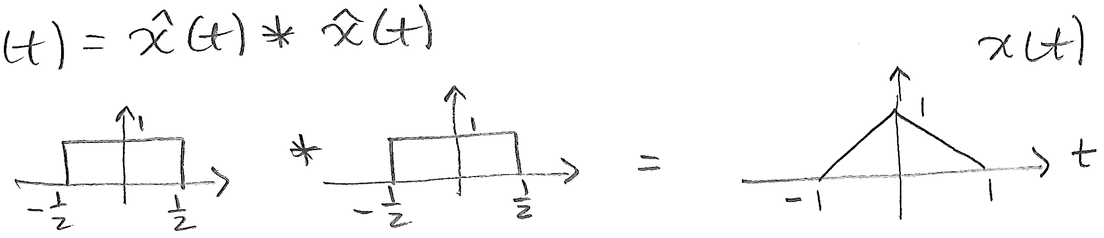
$$X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{1-j\omega+1+j\omega}{1+\omega^2} = \frac{2}{1+\omega^2}$$

(f) Find and sketch $x(t)$, the inverse Fourier transform of $X(j\omega) = \frac{4\sin^2(\omega/2)}{\omega^2}$.

$$X(j\omega) = \frac{2\sin(\omega/2)}{\omega} \cdot \frac{2\sin(\omega/2)}{\omega} = \hat{X}(j\omega)\cdot\hat{X}(j\omega)$$

$$\hat{x}(t) = u(t+1)-u(t-1)$$

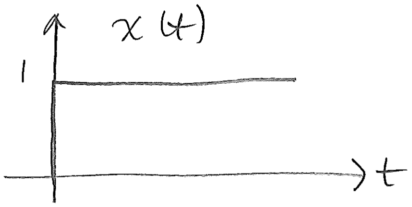
$$x(t) = \hat{x}(t) * \hat{x}(t)$$

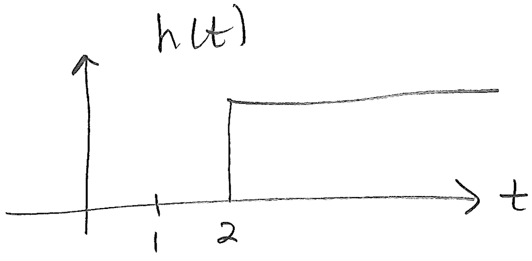


$$x(t) = [1-|t|][u(t+1)-u(t-1)]$$

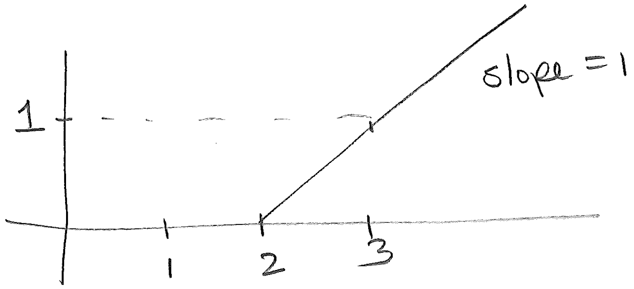
Problem F.2:
(10 points each) Consider the convolution $y(t) = x(t) * h(t)$. For each part, $x(t)$ and $h(t)$ are given. Sketch $x(t)$, $h(t)$, and $y(t)$ in the space provided with correct labels for all axes. You do not have to provide equations for your answers since you may be able to do some or all of these by inspection.

(a) $x(t) = u(t)$ and $h(t) = u(t-2)$

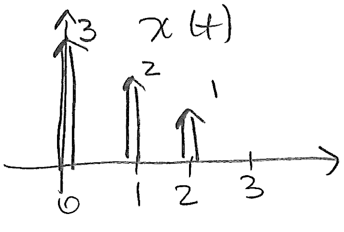


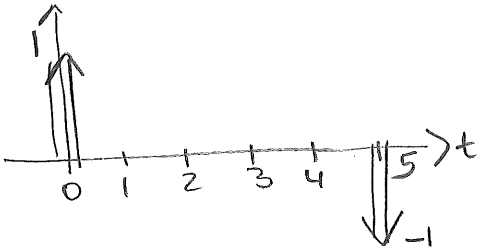


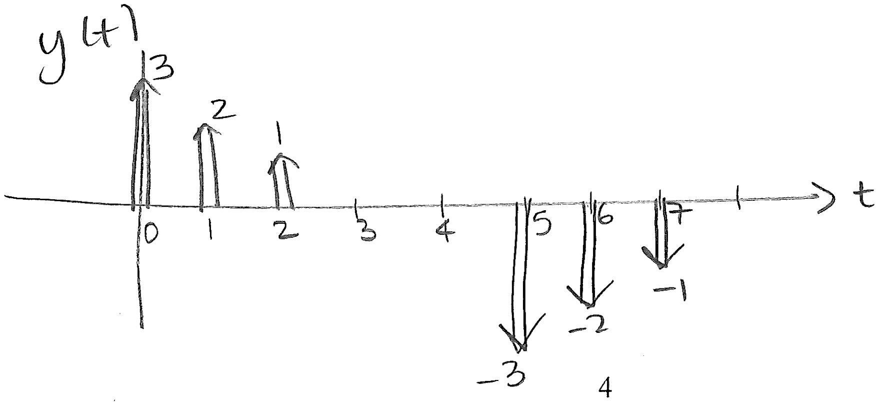
$y(t) = (t-2)u(t-2)$



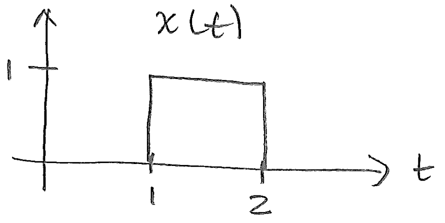
(b) $x(t) = 3\delta(t) + 2\delta(t-1) + \delta(t-2)$ and $h(t) = \delta(t) - \delta(t-5)$

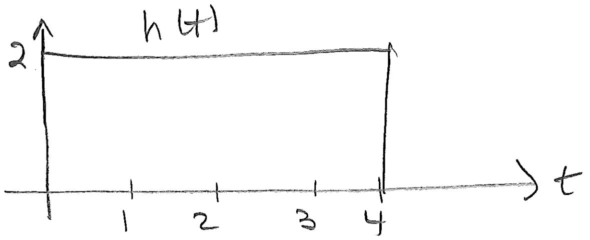


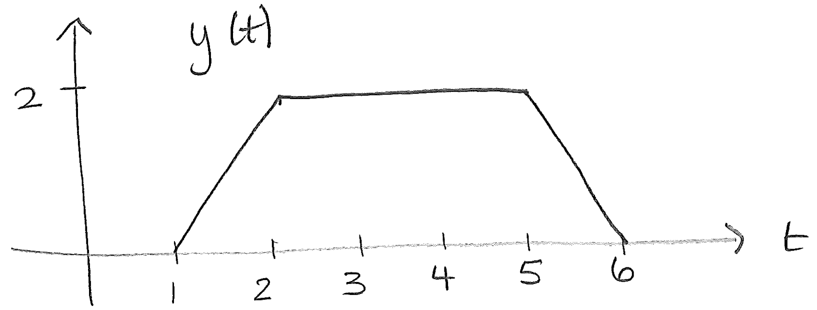




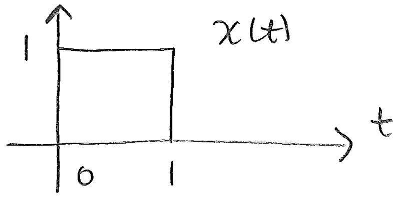
(c) $x(t) = u(t-1) - u(t-2)$ and $h(t) = 2[u(t) - u(t-4)]$

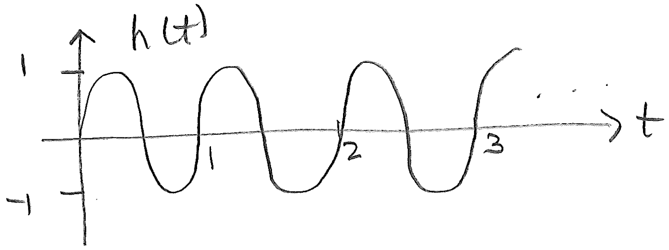


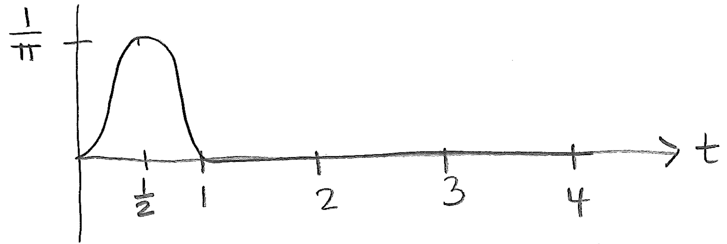




(d) $x(t) = u(t) - u(t-1)$ and $h(t) = \sin(2\pi t)u(t)$







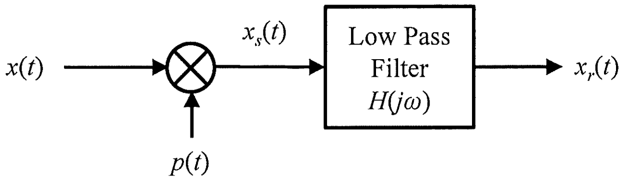
$y(t) = 0$ for $t<0$
and $t > 1$

Max amplitude is at $t = \frac{1}{2}$

$$y(\tfrac{1}{2}) = \int_0^{\frac{1}{2}} \sin(2\pi t)\,dt = \frac{-\cos(2\pi t)}{2\pi}\Bigg|_0^{\frac{1}{2}} = -\frac{1}{2\pi}\Big[\underset{-1}{\cos(\pi)} - \underset{+1}{\cos(0)}\Big] = \frac{1}{\pi}$$

Problem F.3:

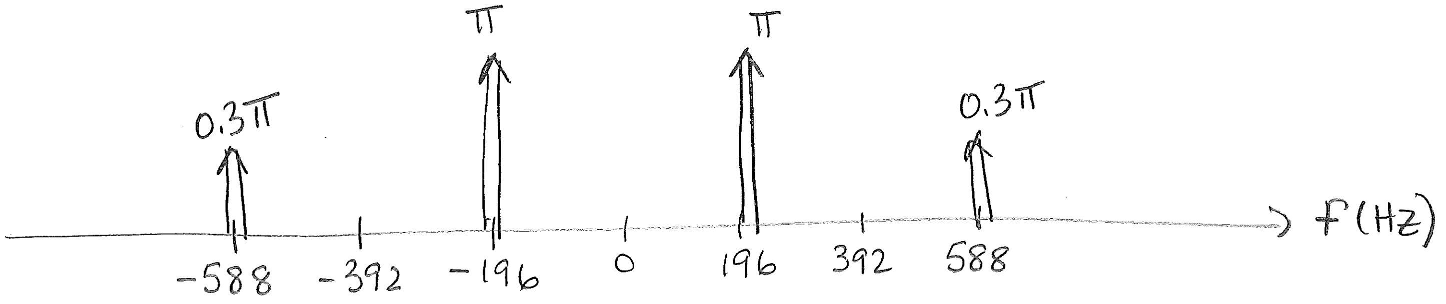
This problem considers sampling and reconstruction as shown in the figure below where $x(t)$ is the signal being sampled, $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ is an impulse train, $H(j\omega)$ is an ideal low pass filter with cutoff frequency $\omega_s/2$ and gain T_s, and $x_r(t)$ is the reconstructed signal. For all parts of this problem the input is the periodic signal $x(t) = \cos(\omega_0 t) + 0.3\cos(3\omega_0 t)$, and the fundamental frequency $f_0 = \omega_0/(2\pi)$ is 196 Hz. This signal is a (poorly) synthesized version of a fictitious musical instrument playing the note "G" below middle "C".



(a) (10 pts) Find and accurately sketch $X(j\omega)$, the Fourier transform of $x(t)$, as a function of $f = \omega/2\pi$.

$x(t) = \cos(\omega_0 t) + 0.3\cos(3\omega_0 t)$

$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) + 0.3\pi\delta(\omega-3\omega_0) + 0.3\pi\delta(\omega+3\omega_0)$



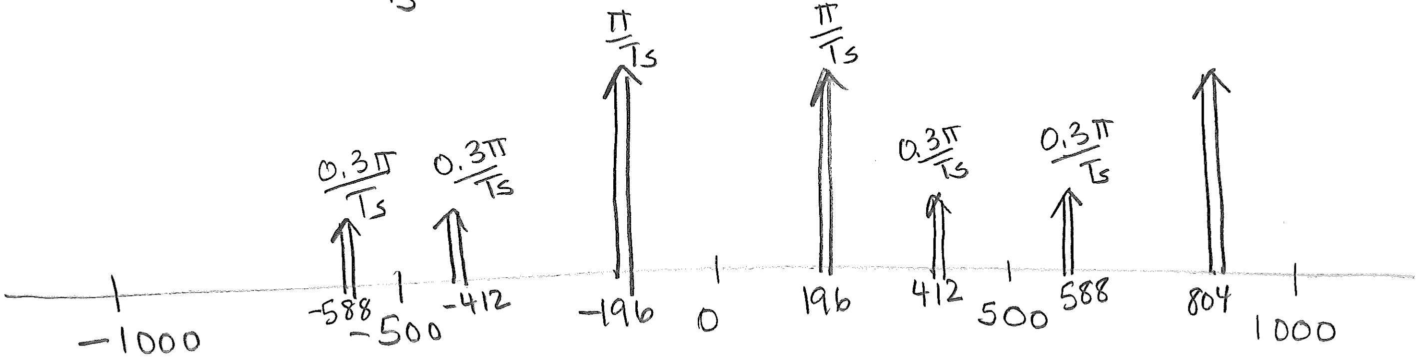
(b) (5 pts) For what choices of sampling frequency f_s will $x_r(t) = x(t)$?

$f_s > 2(588)$

$f_s > 1176$ Hz

(c) (10 pts) Now let the sampling frequency be $f_s = 1000$ Hz. Accurately sketch $X_s(j\omega)$, the Fourier transform of the sampled signal $x_s(t)$, as a function of f for $-1000 \leq f \leq 1000$ Hz. Find an equation for the reconstructed signal $x_r(t)$ for this value of f_s.

X(jω) is replicated every 1000 Hz with a gain of $\frac{1}{T_s}$



$$X_r(j\omega) = 0.3\pi\delta(\omega+2\pi 412) + \pi\delta(\omega+2\pi 196) + \pi\delta(\omega-2\pi 196)$$
$$0.3\pi\delta(\omega-2\pi 412)$$

$$x_r(t) = \cos(2\pi 196 t) + 0.3\cos(2\pi 412 t)$$

(d) (5 pts) Is the reconstructed signal $x_r(t)$ periodic for $f_s = 1000$ Hz? If so, what is its fundamental frequency? If not, explain why not.

2 frequencies: 196 kHz & 412 kHz

412 kHz is not a multiple of 196 kHz

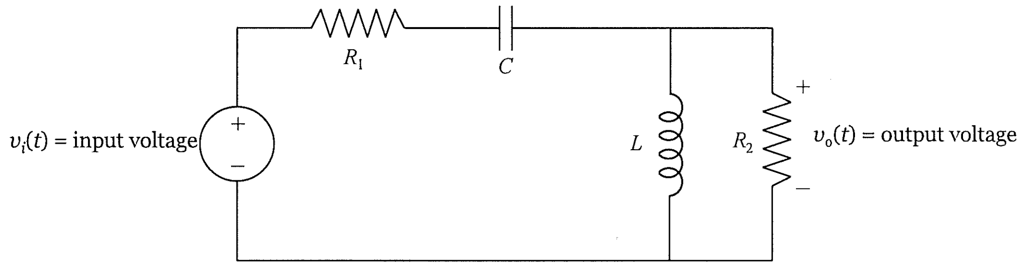
Find greatest common denominator

196 = 2×2×7×7   412 = 2×2×103

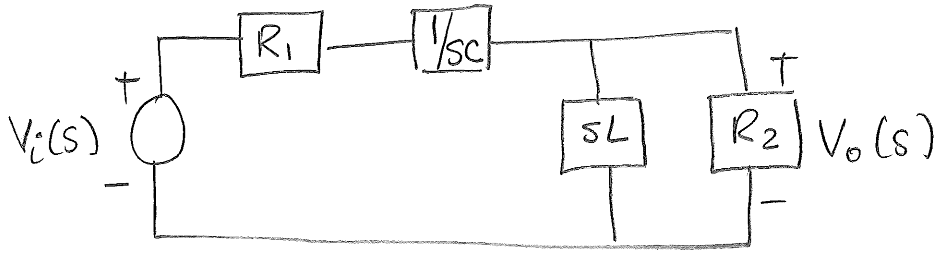
GCD = 4Hz, Yes, $x_r(t)$ is periodic w/ a fundamental frequency of 4 Hz

Problem F.4:

Consider the circuit shown below with input voltage source $v_i(t)$ and output voltage $v_o(t)$:



(a) (5 pts) Draw this circuit in the s-domain assuming zero initial conditions. Properly label all quantities indicated on the original circuit for full credit.



(b) (10 pts) Determine the transfer function relating the input voltage source to the voltage across R_2, $H(s) = V_o(s)/V_i(s)$, by analyzing the circuit in the s-domain. Simplify $H(s)$ so that it is expressed as a ratio of polynomials in s with all like terms combined.

Voltage divider

$$\frac{V_o(s)}{V_i(s)} = \frac{sL \| R_2}{R_1 + \frac{1}{sC} + sL \| R_2} = \frac{\frac{sLR_2}{sL+R_2}}{\frac{1}{sC}(sCR_1+1) + \frac{sLR_2}{sL+R_2}} \times \frac{sC}{sC} \times \frac{sL+R_2}{sL+R_2}$$

$$= \frac{s^2LCR_2}{(sCR_1+1)(sL+R_2) + s^2LCR_2}$$

$$H(s) = \frac{s^2LCR_2}{s^2LC(R_1+R_2) + s(L+CR_1R_2) + R_2}$$

high pass filter

(c) (5 pts) Let $v_i(t) = V_0u(t)$. Use the Final Value Theorem to find the steady state output voltage. Make sure that this value makes sense!

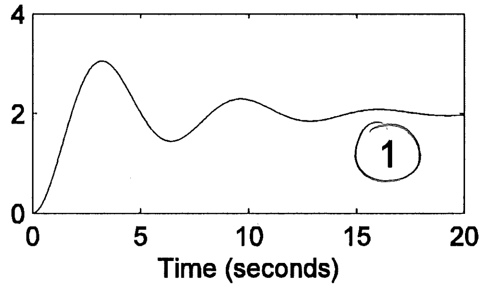
$$Y(s) = \frac{V_0 H(s)}{s} \qquad y_{ss} = sY(s)\Big|_{s=0} = V_0 H(0) = 0$$

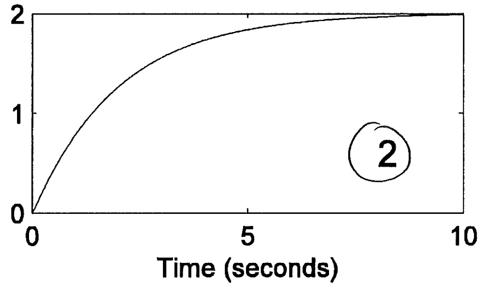
makes sense for a HPF

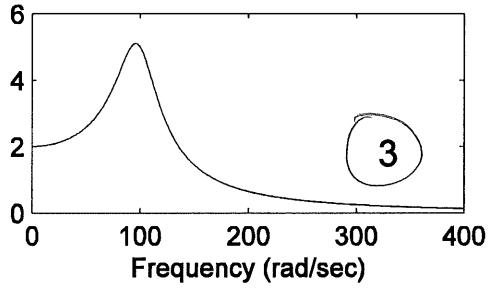
Problem F.5:

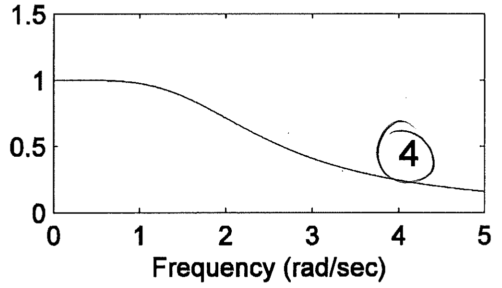
(5 pts each) Consider the six low pass filters whose transfer functions are given below. Your job is to select either a step response (graphs 1 and 2) or a frequency response (graphs 3 through 6) from the numbered choices below that matches the transfer function. Each one will have exactly one match (that is, there will be either a step response or a frequency response that matches, but not both). Put the number of the matching graph in the space provided.

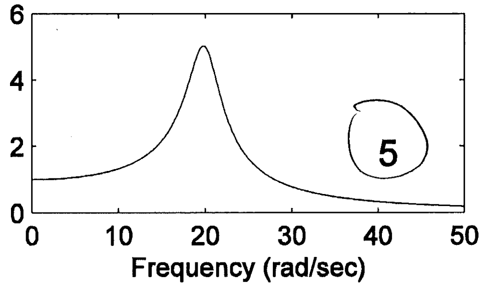
	Transfer Function	Graph	Transfer Function	Graph	
$\omega_n=100$ $\zeta=0.2$	(a) $H(s) = \dfrac{20000}{s^2+40s+10000}$	3	(d) $H(s) = \dfrac{2}{s^2+0.4s+1}$	1	$\omega_n=1$ $\zeta = 0.2$
$\omega_n=5$ $\zeta=5$	(b) $H(s) = \dfrac{50}{s^2+50s+25}$	2	(e) $H(s) = \dfrac{4}{s^2+2.8s+4}$	4	$\omega_n=2$ $\zeta = 0.7$
$\omega_n=1$ $\zeta=1$	(c) $H(s) = \dfrac{2}{s^2+2s+1}$	6	(f) $H(s) = \dfrac{400}{s^2+4s+400}$	5	$\omega_n=20$ $\zeta = 0.1$

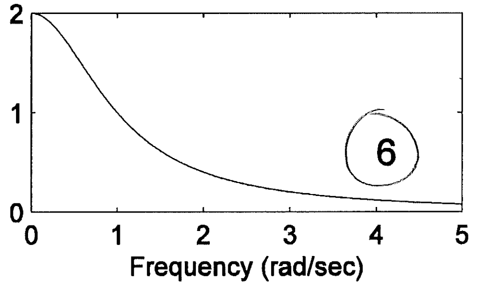






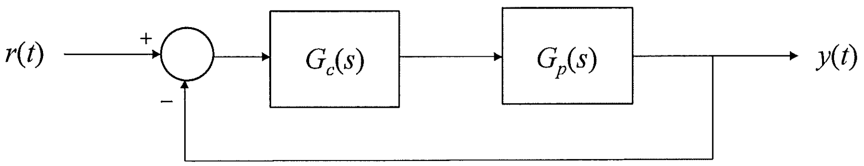






Problem F.6:

This problem considers control of a plant $G_p(s) = \frac{s-b}{s^2+6s+8}$ using a proportional controller $G_c(s) = K_p$.



(a) (10 pts) Find the transfer function $H(s)$ of the closed loop system in terms of b, the plant zero location, and K_p, the proportional gain. Express your answer as a ratio of polynomials in s with all like terms combined.

$$H(s) = \frac{\frac{K_p(s-b)}{s^2+6s+8}}{1+\frac{K_p(s-b)}{s^2+6s+8}} = \frac{K_p(s-b)}{s^2+(6+K_p)s+8-bK_p}$$

(b) (5 pts) For $b = 2$, find all $K_p > 0$ for which the closed loop system is stable.

must have $8-2K_p > 0$

$$\boxed{K_p < 4}$$

(c) (10 pts) Find the complete step response, $y(t)$ for $r(t) = u(t)$, for $b = 2$ and $K_p = 1$. Comment on the ability of the closed loop system to track a step input.

$$H(s) = \frac{s-2}{s^2+7s+6}$$

$$Y(s) = \frac{s-2}{s(s+1)(s+6)} = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+6}$$

$c_1 = -\frac{1}{3}$

$c_2 = \frac{-1-2}{(-1)(5)} = \frac{3}{5}$

$c_3 = \frac{-6-2}{(-6)(-5)} = -\frac{8}{30} = -\frac{4}{15}$

$$y(t) = -\frac{1}{3}u(t) + \frac{3}{5}e^{-t}u(t) - \frac{4}{15}e^{-6t}u(t)$$

It doesn't track a step at all (wrong sign)

This problem was porrly designed. Full credit was given for solution Ⓐ on this page or Solution Ⓑ on the next page.

Problem F.7:

Solution Ⓐ

Consider the same feedback system of the previous problem with $G_p(s) = \dfrac{s+2}{s^2+9s+14}$ and a PI controller, $G_c(s) = K_p + K_i/s$.

(a) (10 pts) Find the transfer function $H(s)$ of the closed loop system in terms of the controller gains, K_p and K_i. Express your answer as a ratio of polynomials in s with all like terms combined.

$$H(s) = \frac{\dfrac{(K_p s + K_i)(s+2)}{s(s^2+9s+14)}}{1 + \dfrac{(K_p s + K_i)(s+2)}{s(s^2+9s+14)}} = \frac{K_p s^2 + (2K_p + K_i)s + 2K_i}{s^3 + (9+K_p)s^2 + (14+2K_p+K_i)s + 2K_i}$$

(b) (5 pts) What are the restrictions, if any, on K_p and K_i for the system to achieve perfect tracking of a step input in the steady-state?

$$y_{ss} = sY(s)\big|_{s=0} = H(0) = \frac{2K_i}{2K_i} = 1$$

The system will perfectly track a step in the steady state as long as K_p and K_i are selected to make the system stable.

(c) (10 pts) Suppose you want the closed loop system to have one pole at $s = -10$ and two repeated poles at $s = -2$. Either find $K_p > 0$ and $K_i > 0$ that will achieve this goal, or show that it can't be done.

match denominators

$$(s+10)(s+2)^2 = (s+10)(s^2+4s+4) = s^3 + 4s^2 + 4s + 10s^2 + 40s + 40$$

$$= s^3 + 14s^2 + 44s + 40$$

s^2: $9 + K_p = 14$

$K_p = 5$

s^0: $2K_i = 40$

$K_i = 20$

$\rightarrow$ check s^1

$14 + 2(5) + 20 =$

$14 + 30 = 44$

which agrees with what we want

$\boxed{K_p = 5 \text{ \& } K_i = 20}$

Solution $\boxed{B}$

Problem F.7:

Consider the same feedback system of the previous problem with $G_p(s) = \dfrac{s+2}{s^2+9s+14}$ and a PI controller, $G_c(s) = K_p + K_i/s$.

(a) (10 pts) Find the transfer function $H(s)$ of the closed loop system in terms of the controller gains, K_p and K_i. Express your answer as a ratio of polynomials in s with all like terms combined.

$$G_p(s) = \frac{s+2}{(s+2)(s+7)} = \frac{1}{s+7}$$

$$H(s) = \frac{\dfrac{K_p s + K_i}{s(s+7)}}{1 + \dfrac{K_p s + K_i}{s(s+7)}} = \frac{K_p s + K_i}{s^2 + (7+K_p)s + K_i}$$

(b) (5 pts) What are the restrictions, if any, on K_p and K_i for the system to achieve perfect tracking of a step input in the steady-state?

$$y_{ss} = sY(s)\big|_{s=0} = H(0) = \frac{K_i}{K_i} = 1$$

The system will perfectly track a step as long as the system is stable; $K_p > -7$ and $K_i > 0$

(c) (10 pts) Suppose you want the closed loop system to have one pole at $s = -10$ and two repeated poles at $s = -2$. Either find $K_p > 0$ and $K_i > 0$ that will achieve this goal, or show that it can't be done.

Match denominators

$$(s+10)(s+2)^2 = (s+10)(s+2)^2 = s^3 + 14s^2 + 44s + 40$$

cubic

Denominator of $H(s)$ is a quadratic and thus only has 2 poles $\Rightarrow$ can't be done