

**GEORGIA INSTITUTE OF TECHNOLOGY**  
SCHOOL of ELECTRICAL & COMPUTER ENGINEERING  
**FINAL EXAM**

DATE: 29-Apr-13

COURSE: ECE 3084A (Prof. Michaels)

NAME: \_\_\_\_\_  
                    LAST,                    FIRST

STUDENT #: \_\_\_\_\_

- Write your name on the front page **ONLY**. Do not unstaple the test. You should have an additional sheet with Fourier transform tables on one side and Laplace transform tables on the other.
- No calculators, laptops, phones, or other electronic devices allowed. Keep the tables clear of all back backs, books, etc.
- Closed book. However, two pages ( $8\frac{1}{2}'' \times 11''$ ) of **HAND-WRITTEN** notes permitted. OK to write on both sides.
- Unless stated otherwise, justify your reasoning clearly to receive any partial credit.
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.
- The room is small for the number of students in this section. **BE CAREFUL TO NOT LET YOUR EYES WANDER.** Any sort of communication with your fellow students during this exam is strictly forbidden. Any attempt to read off of your neighbor's exam will result in unpleasant disciplinary action.
- Good luck!

<i>Problem</i>	<i>Value</i>	<i>Score</i>
1	40	
2	20	
3	25	
4	35	
5	30	
6	50	
Total	200	

**Problem F.1:**

Short problem assortment (5 pts each). The eight parts of this problem are unrelated to each other. Make sure that your answers are simplified as much as possible unless stated otherwise.

(a) Evaluate and simplify the following expression:

$$x(t) = \int_{-\infty}^t \left\{ \sum_{k=1}^3 \tau^{4-k} \delta(\tau - 2k) \right\} d\tau =$$

(b) Evaluate and simplify the following expression:

$$x(t) = \frac{d}{dt} \{t^2[u(t-2) - u(t+2)]\} =$$

(c) Evaluate  $y(t) = x(t) * h(t)$  where  $x(t) = \delta(t-1) - \delta(t-3)$  and  $h(t) = e^{-5t}u(t)$ .

(d) Find  $X(j\omega)$ , the Fourier transform of the signal  $x(t) = e^{-t}[u(t) - u(t-2)]$

(e) Find  $x(t)$ , the inverse Fourier transform of  $X(j\omega) = \sum_{k=-1}^1 \frac{(-1)^k}{1+jk} \delta(\omega - 2\pi k/5)$ .

(f) Find  $X(s)$ , the Laplace transform of  $x(t) = t^2 \cos(5t)u(t)$ . You do not need to simplify your answer.

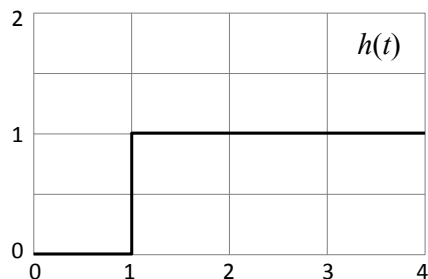
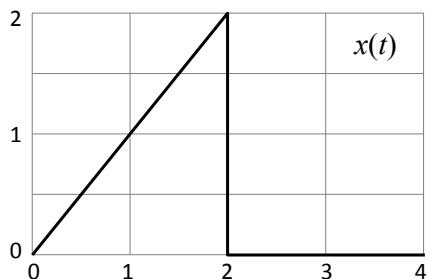
(g) Find  $x(t)$ , the inverse Laplace transform of  $X(s) = \frac{s+5}{s^2+2s+5}$

(h) Consider the following differential equation with input (forcing function)  $x(t)$  and output  $y(t)$ . Find the transfer function of the system described by this differential equation.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - 2x(t)$$

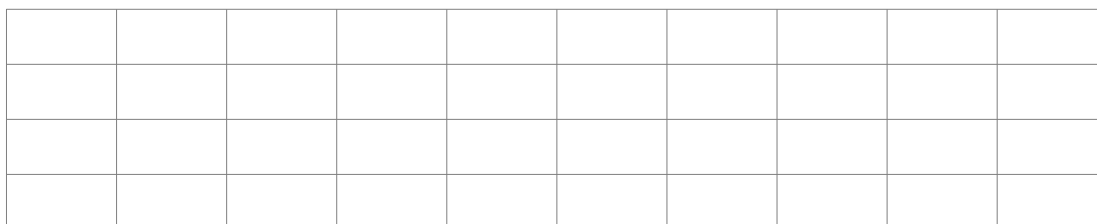
**Problem F.2:**

Consider the convolution of the following two signals,  $x(t)$  and  $h(t)$ :

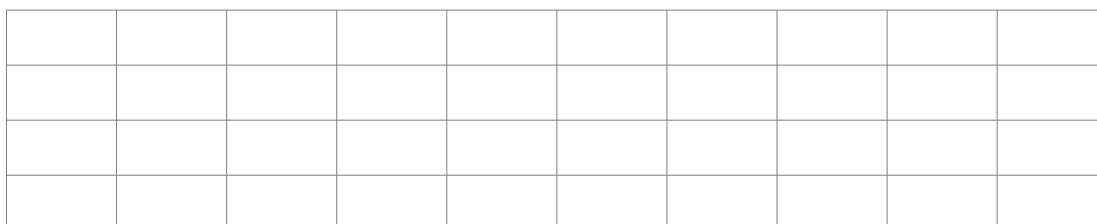


(a) (5 points) Write an equation for each of the above signals.

(b) (5 points) Plot  $x(\tau)$  and  $h(t - \tau)$  on the graph below for an arbitrary value of  $t < 0$ . Plot both signals on the same time axis. Clearly identify each function. Carefully label both axes and all points of interest on  $h(t - \tau)$ .

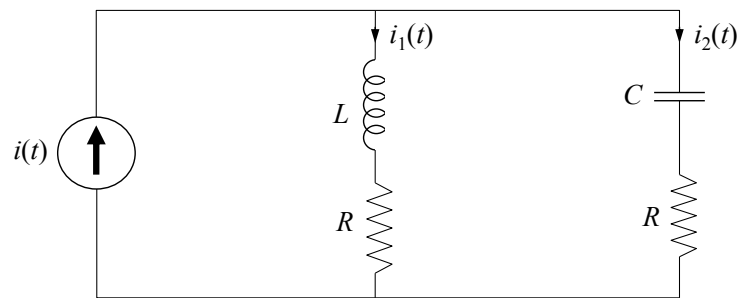


(c) (10 points) Calculate  $y(t) = x(t) * h(t)$  and plot it on the graph below. Clearly label both axes. Show all work for full credit.

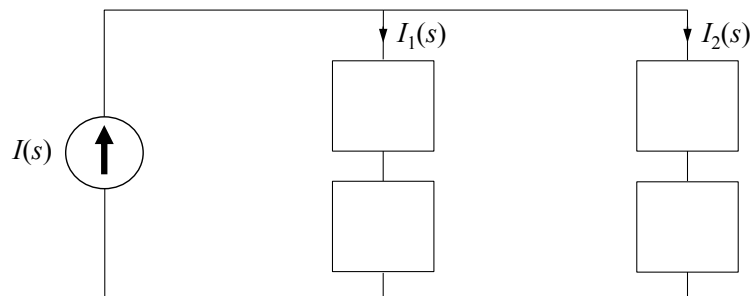


**Problem F.3:**

Consider the circuit shown below with input current source  $i(t)$  and output current  $i_2(t)$ :



- (a) (5 pts) In the drawing below, insert the impedance of each circuit element into its corresponding block.



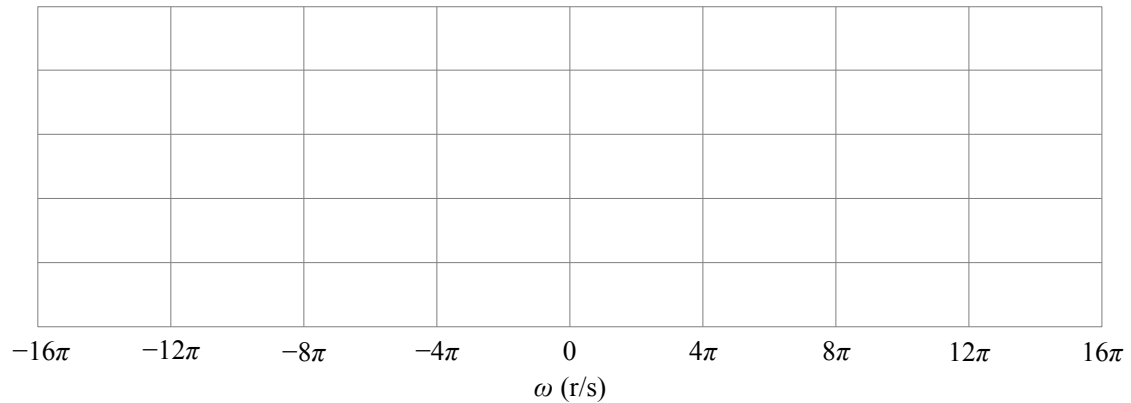
- (b) (15 pts) Determine the transfer function of the circuit,  $H(s) = I_2(s)/I(s)$ , by analyzing it in the  $s$ -domain. Express  $H(s)$  as a ratio of polynomials.

- (c) (5 pts) Let  $i(t) = I_0u(t)$ . Use the Final Value Theorem to find the steady state output current.

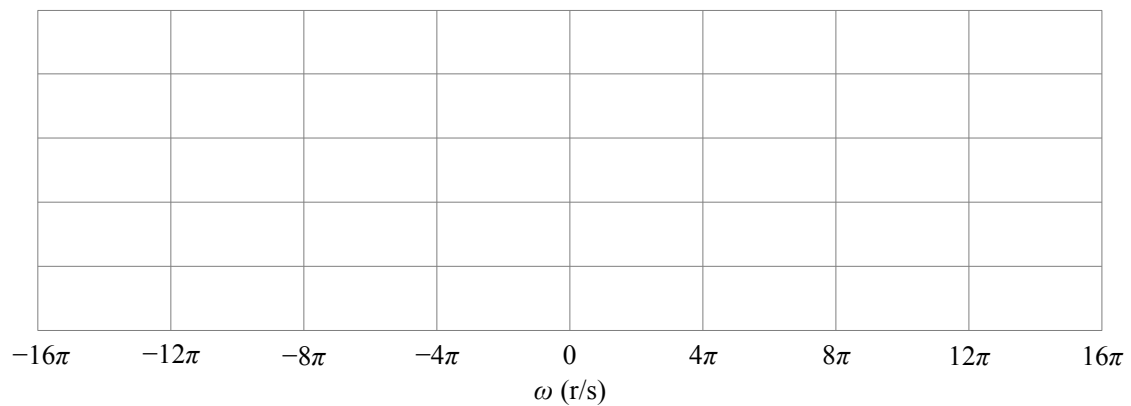
**Problem F.4:**

In this problem we consider the effect of windowing a signal in the time domain. Since most signals are finite in length, some type of time domain windowing is usually unavoidable.

- (a) (5 pts) Assume that  $x(t)$ , the signal to be windowed, is a 4 Hz sinusoid;  $x(t) = \cos(2\pi 4t)$ . What is  $X(j\omega)$ , the Fourier transform of the sinusoid  $x(t)$ ? Plot it on the graph below. Be sure to label all amplitudes.

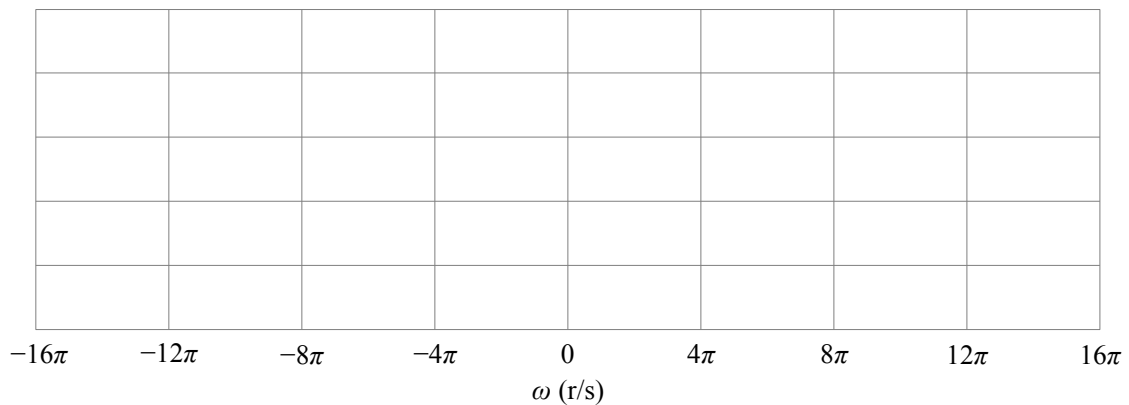


- (b) (10 pts) Consider the rectangular window function,  $w(t) = u(t + T/2) - u(t - T/2)$ . What is  $W(j\omega)$ , the Fourier transform of the window function? Accurately sketch its magnitude,  $|W(j\omega)|$ , on the graph below for  $T = 1$ s. Be sure to label all important amplitudes.



- (c) (5 pts) For any window function  $w(t)$ , the windowed signal is  $y(t) = x(t)w(t)$ . Express  $Y(j\omega)$ , the Fourier transform of  $y(t)$ , in terms of  $W(j\omega)$ , the Fourier transform of  $w(t)$ , for  $x(t) = \cos(2\pi 4t)$ .

- (d) (10 pts) For  $x(t) = \cos(2\pi 4t)$ , the windowed signal is  $y(t) = \cos(2\pi 4t)w(t)$ . Find  $Y(j\omega)$  and sketch its magnitude,  $|Y(j\omega)|$ , below. Be sure to label all important amplitudes.

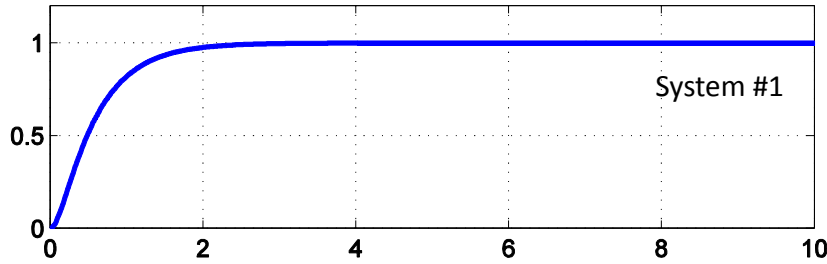


- (e) (5 pts) One goal of windowing a long signal is to use the shorter signal,  $y(t)$ , to estimate the frequency content of the longer signal,  $x(t)$ ; that is, we want  $Y(j\omega)$  to approximate  $X(j\omega)$ . Would a window length of  $T = 2$ s do a better job of estimating  $X(j\omega)$  as compared to the original window of 1s? Why or why not?

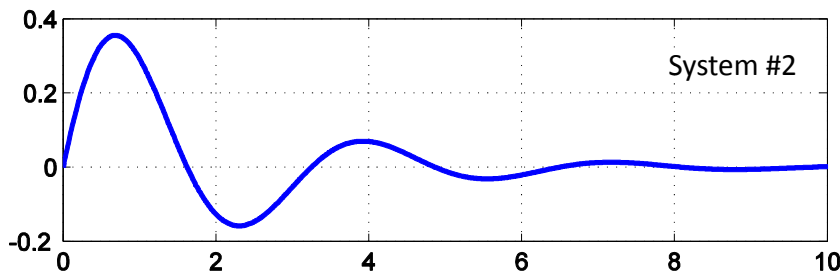
**Problem F.5:**

The two parts of this problem are unrelated.

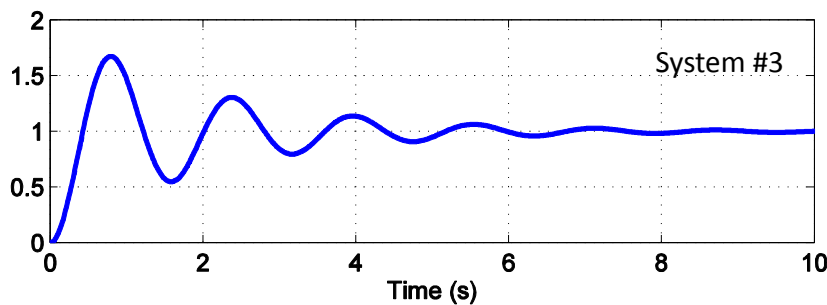
- (a) (5 pts each) Consider three LTI systems, all with second order transfer functions of the form  $H(s) = \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2}$  (the coefficients are different for each system). A unit step is input to each system, and the three step responses shown below are measured (Systems #1, #2 and #3). For each system select a transfer function from the six provided below that is consistent with the measured step response. Put the letter (A-F) of the transfer function in the box to the right of the step response.



Transfer  
Function



Transfer  
Function



Transfer  
Function

A.  $H(s) = \frac{2}{s^2 + 4s + 4}$

B.  $H(s) = \frac{16}{s^2 + 10s + 16}$

C.  $H(s) = \frac{16}{s^2 + s + 16}$

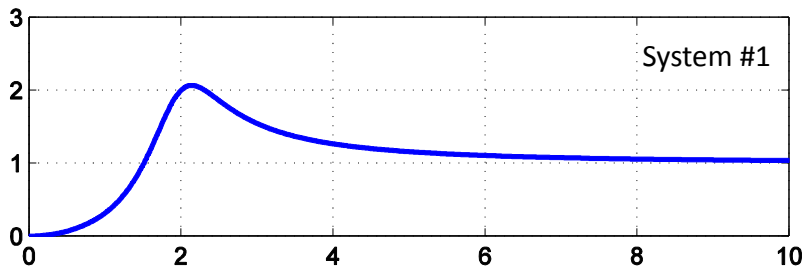
D.  $H(s) = \frac{s}{s^2 + s + 4}$

E.  $H(s) = \frac{2s}{s^2 + s + 16}$

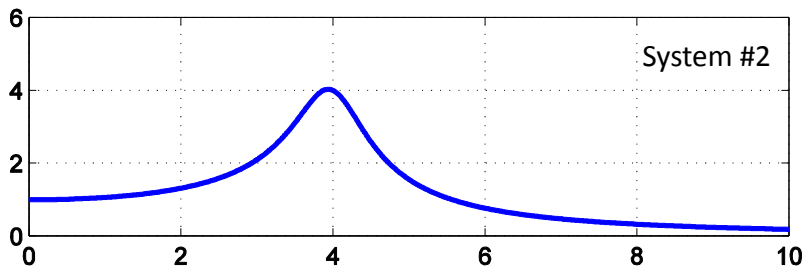
F.  $H(s) = \frac{s^2}{s^2 + 3s + 4}$



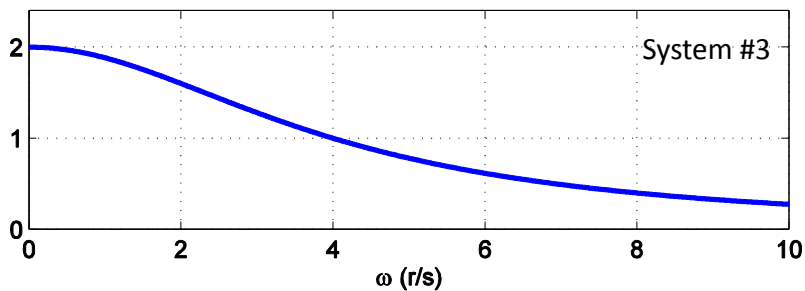
- (b) (5 pts each) Let's again consider three LTI systems, which may or may not be the same as the ones in part (a), all with second order transfer functions of the form  $H(s) = \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2}$  (the coefficients are different for each system). The three frequency responses of the systems are shown below (Systems #1, #2 and #3). For each system select a transfer function from the six provided below that is consistent with the frequency response shown. Put the letter (A-F) of the transfer function in the box to the right of the frequency response.



Transfer  
Function



Transfer  
Function



Transfer  
Function

A.  $H(s) = \frac{32}{s^2 + 8s + 16}$

B.  $H(s) = \frac{16}{s^2 + s + 16}$

C.  $H(s) = \frac{1}{s^2 + s + 4}$

D.  $H(s) = \frac{8s}{s^2 + 8s + 16}$

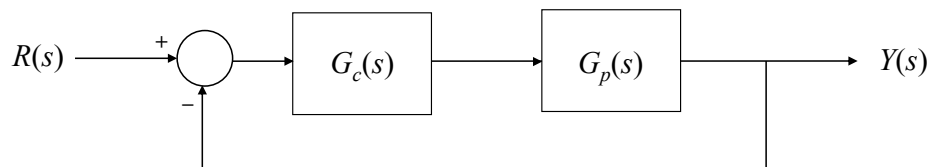
E.  $H(s) = \frac{8s^2}{s^2 + 8s + 16}$

F.  $H(s) = \frac{s^2}{s^2 + s + 4}$

**Problem F.6:**

Consider the feedback control system shown below for the following plant with two poles,

$$G_p(s) = \frac{1}{(s+1)(s+7)}.$$



For parts (a), (b) and (c), use a proportional controller,  $G_c(s) = K_p$ .

(a) (5 pts) Find the transfer function  $H(s)$  of the closed loop system in terms of  $K_p$ . Express your answer as a ratio of polynomials with all like terms combined.

(b) (10 pts) For  $K_p = 5$ , find the complete step response of the overall closed loop system. That is, find  $y(t)$  for  $r(t) = u(t)$ .

(c) (5 pts) What value of  $K_p$  results in a critically damped system?

- (d) (10 pts) Increasing  $K_p$  from the value found in part (c) will (eventually) have which of the following effects: (circle True or False for each statement):

The steady state error for tracking a step will increase	True	False
The system will begin to oscillate	True	False
The damping ratio $\zeta$ will increase	True	False
The natural frequency $\omega_n$ will increase	True	False
The system will become unstable	True	False

For parts (e) and (f), use a PI controller,  $G_c(s) = K_p + K_i/s$ .

- (e) (10 pts) Find the transfer function  $H(s)$  of the closed loop system in terms of  $K_p$  and  $K_i$ . Express your answer as a ratio of polynomials with all like terms combined.

- (f) (10 pts) Your  $H(s)$  of part (d) should have three poles. However, a PI controller only has two parameters, so you cannot place all three poles arbitrarily. Suppose you specify that one of the poles is at  $s = -2$  and that the other two poles are repeated (i.e., both are at the same unknown location). Find  $K_p$ ,  $K_i$ , and the unknown location of the repeated pole.

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NAME: Answer Key  
LAST, FIRST

STUDENT #: \_\_\_\_\_

- 
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**Problem F.1:**

Short problem assortment (5 pts each). The eight parts of this problem are unrelated to each other. Make sure that your answers are simplified as much as possible unless stated otherwise.

(a) Evaluate and simplify the following expression:

$$\begin{aligned}
 x(t) &= \int_{-\infty}^t \left\{ \sum_{k=1}^3 \tau^{4-k} \delta(\tau - 2k) \right\} d\tau = \int_{-\infty}^t [\tau^3 \delta(\tau - 2) + \tau^2 \delta(\tau - 4) + \tau \delta(\tau - 6)] d\tau \\
 &= \int_{-\infty}^t [8\delta(\tau - 2) + 16\delta(\tau - 4) + 6\delta(\tau - 6)] d\tau \\
 &= 8u(t - 2) + 16u(t - 4) + 6u(t - 6)
 \end{aligned}$$

(b) Evaluate and simplify the following expression:

$$\begin{aligned}
 x(t) &= \frac{d}{dt} \{ t^2 [u(t - 2) - u(t + 2)] \} = t^2 [\delta(t - 2) - \delta(t + 2)] + 2t [u(t - 2) - u(t + 2)] \\
 &= 4\delta(t - 2) - 4\delta(t + 2) + 2t [u(t - 2) - u(t + 2)]
 \end{aligned}$$

(c) Evaluate  $y(t) = x(t) * h(t)$  where  $x(t) = \delta(t - 1) - \delta(t - 3)$  and  $h(t) = e^{-5t}u(t)$ .

$$y(t) = e^{-5(t-1)}u(t-1) - e^{-5(t-3)}u(t-3)$$

(d) Find  $X(j\omega)$ , the Fourier transform of the signal  $x(t) = e^{-t}[u(t) - u(t - 2)]$

$$\begin{aligned}
 x(t) &= e^{-t}u(t) - e^{-(t-2)}u(t-2)e^{-2} \\
 X(j\omega) &= \frac{1}{1+j\omega} - \frac{e^{-2}e^{-j2\omega}}{1+j\omega} \\
 &= \frac{1 - e^{-2(1+j\omega)}}{1+j\omega}
 \end{aligned}$$

(e) Find  $x(t)$ , the inverse Fourier transform of  $X(j\omega) = \sum_{k=-1}^1 \frac{(-1)^k}{1+jk} \delta(\omega - 2\pi k/5)$ .

$$x(t) = \frac{1}{2\pi} \sum_{k=-1}^1 \frac{(-1)^k}{1+jk} e^{j\frac{2\pi k}{5}t}$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi} \frac{-1}{1+j} e^{j\frac{2\pi}{5}t} + \frac{1}{2\pi} \frac{-1}{1-j} e^{-j\frac{2\pi}{5}t}$$

$$= \frac{1}{2\pi} + \frac{1}{\pi\sqrt{2}} \cos\left(\frac{2\pi t}{5} + \frac{3\pi}{4}\right)$$

$$\left\{ \begin{aligned} \frac{-1}{1+j} &= \frac{-1(1-j)}{(1+j)(1-j)} \\ &= \frac{-1+j}{2} \\ &= \frac{1}{\sqrt{2}} \angle \frac{3\pi}{4} \end{aligned} \right.$$

(f) Find  $X(s)$ , the Laplace transform of  $x(t) = t^2 \cos(5t)u(t)$ . You do not need to simplify your answer.

$t^2 \leftrightarrow \frac{2}{s^3}$  Use modulation property

$$X(s) = \frac{1}{2} \left[ \frac{2}{(s+j5)^3} + \frac{2}{(s-j5)^3} \right]$$

$$= \frac{1}{(s+j5)^3} + \frac{1}{(s-j5)^3}$$

(g) Find  $x(t)$ , the inverse Laplace transform of  $X(s) = \frac{s+5}{s^2+2s+5}$

$$X(s) = \frac{s+5}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4}$$

$$x(t) = e^{-t} \cos(2t) u(t) + 2e^{-t} \sin(2t) u(t)$$

(h) Consider the following differential equation with input (forcing function)  $x(t)$  and output  $y(t)$ . Find the transfer function of the system described by this differential equation.

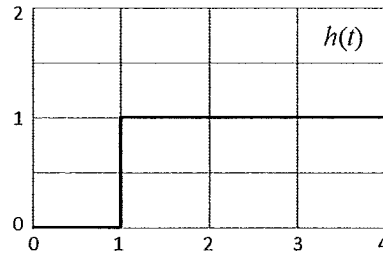
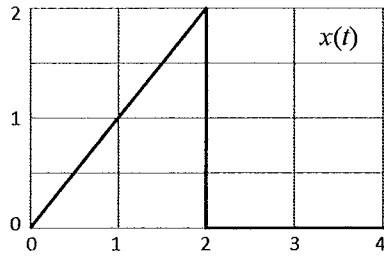
$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - 2x(t)$$

$$(s^2 + 5s + 2)Y(s) = (s - 2)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-2}{s^2+5s+2}$$

**Problem F.2:**

Consider the convolution of the following two signals,  $x(t)$  and  $h(t)$ :

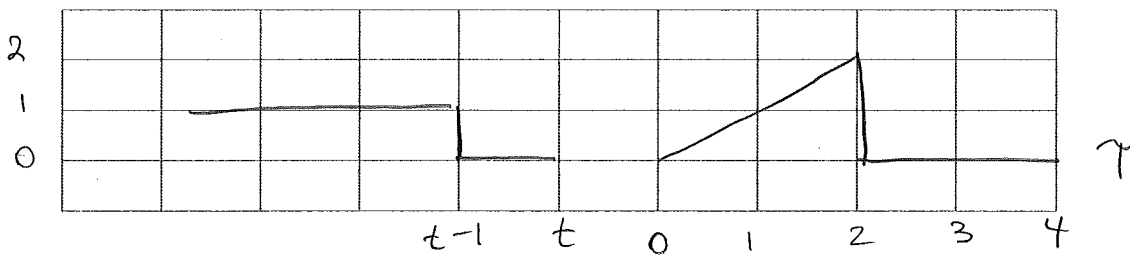


- (a) (5 points) Write an equation for each of the above signals.

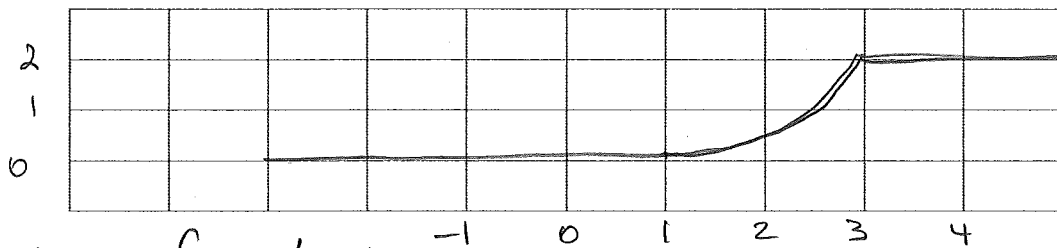
$$x(t) = t[u(t) - u(t-2)]$$

$$h(t) = u(t-1)$$

- (b) (5 points) Plot  $x(\tau)$  and  $h(t-\tau)$  on the graph below for an arbitrary value of  $t < 0$ . Plot both signals on the same time axis. Clearly identify each function. Carefully label both axes and all points of interest on  $h(t-\tau)$ .



- (c) (10 points) Calculate  $y(t) = x(t) * h(t)$  and plot it on the graph below. Clearly label both axes. Show all work for full credit.



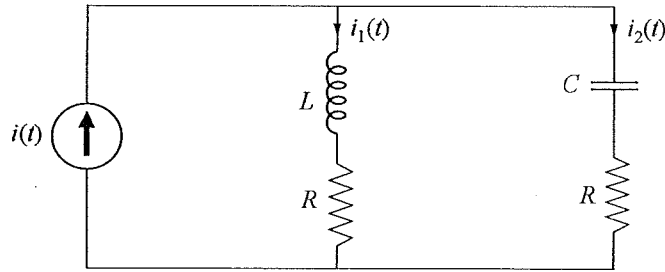
$$y(t) = 0 \text{ for } t < 1$$

$$y(t) = 2 \text{ for } t > 3$$

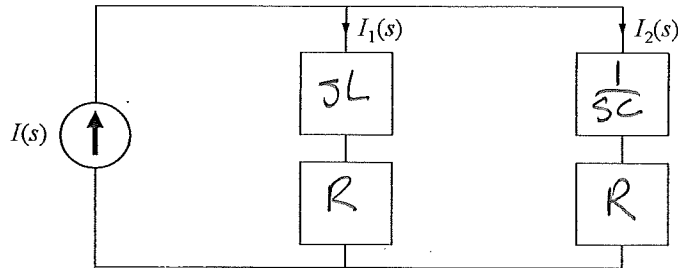
$$\text{for } 1 < t < 3, \quad y(t) = \int_0^{t-1} \tau d\tau = \frac{\tau^2}{2} \Big|_0^{t-1} = \frac{(t-1)^2}{2}$$

**Problem F.3:**

Consider the circuit shown below with input current source  $i(t)$  and output current  $i_2(t)$ :



- (a) (5 pts) In the drawing below, insert the impedance of each circuit element into its corresponding block.



- (b) (15 pts) Determine the transfer function of the circuit,  $H(s) = I_2(s)/I(s)$ , by analyzing it in the  $s$ -domain. Express  $H(s)$  as a ratio of polynomials.

Current divider

$$I_2(s) = I(s) \frac{Z_1}{Z_1 + Z_2}$$

$$H(s) = \frac{Z_1}{Z_1 + Z_2} = \frac{sL + R}{sL + R + \frac{1}{sC} + R} = \frac{s^2 LC + sRC}{s^2 LC + 2sRC + 1}$$

- (c) (5 pts) Let  $i(t) = I_0 u(t)$ . Use the Final Value Theorem to find the steady state output current.

$$I_2(s) = I(s) H(s) = \frac{LCs^2 + RCS}{s(LCs^2 + 2RCS + 1)}$$

$$i_{2ss} = \lim_{s \rightarrow 0} s I_2(s) = 0$$

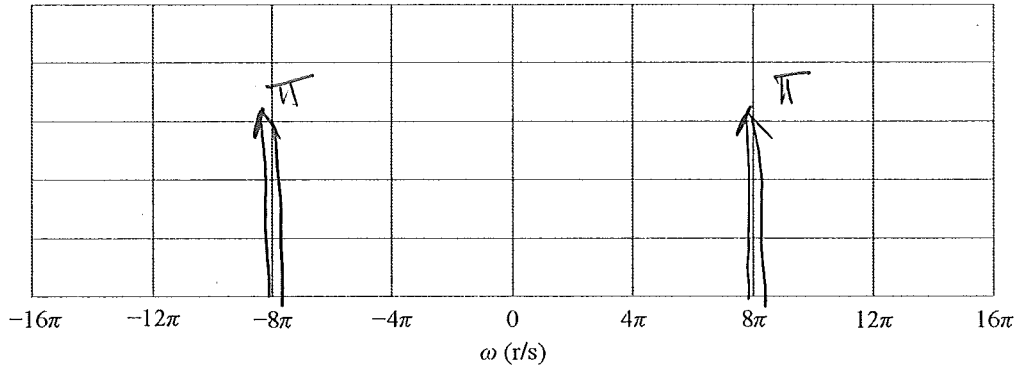


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In this problem we consider the effect of windowing a signal in the time domain. Since most signals are finite in length, some type of time domain windowing is usually unavoidable.

- (a) (5 pts) Assume that  $x(t)$ , the signal to be windowed, is a 4 Hz sinusoid;  $x(t) = \cos(2\pi 4t)$ . What is  $X(j\omega)$ , the Fourier transform of the sinusoid  $x(t)$ ? Plot it on the graph below. Be sure to label all amplitudes.

$$X(j\omega) = \pi \delta(\omega - 8\pi) + \pi \delta(\omega + 8\pi)$$



- (b) (10 pts) Consider the rectangular window function,  $w(t) = u(t + T/2) - u(t - T/2)$ . What is  $W(j\omega)$ , the Fourier transform of the window function? Accurately sketch its magnitude,  $|W(j\omega)|$ , on the graph below for  $T = 1$ s. Be sure to label all important amplitudes.

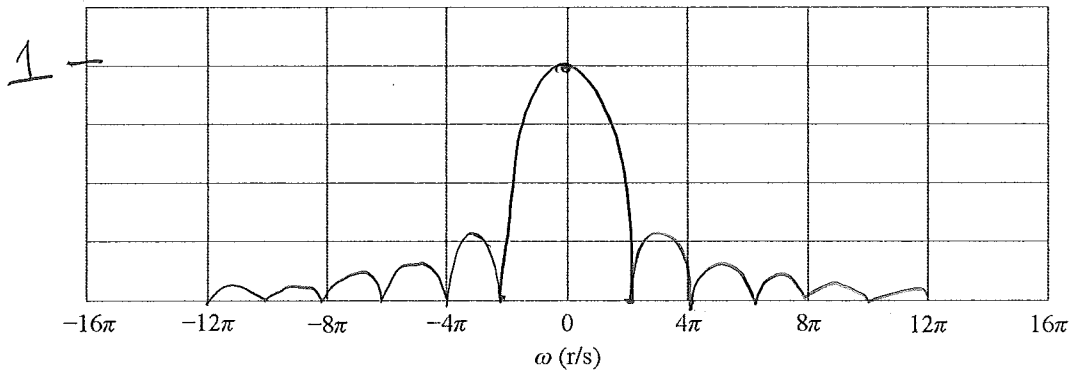
$$W(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

for  $T = 1$ ,

$$W(j\omega) = \frac{\sin(\omega/2)}{\omega/2}$$

Zeros at  $\frac{\omega}{2} = n\pi$   
 $\omega = 2n\pi$

$$W(j0) = \frac{\omega/2}{\omega/2} = 1$$



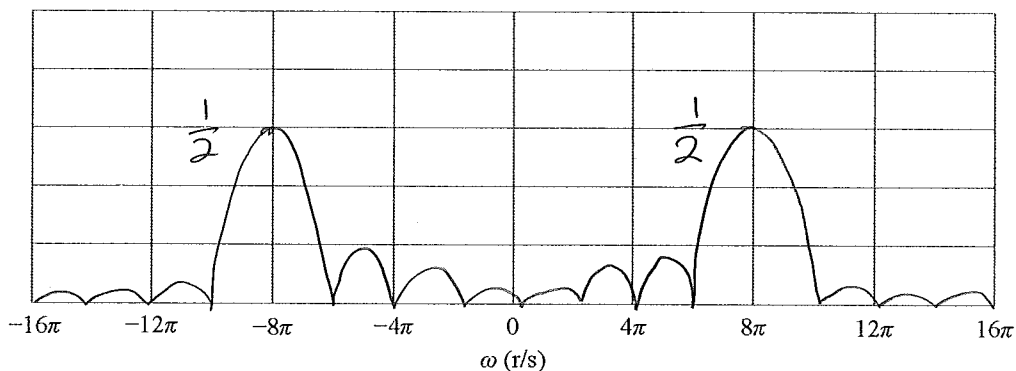
- (c) (5 pts) For any window function  $w(t)$ , the windowed signal is  $y(t) = x(t)w(t)$ . Express  $Y(j\omega)$ , the Fourier transform of  $y(t)$ , in terms of  $W(j\omega)$ , the Fourier transform of  $w(t)$ , for  $x(t) = \cos(2\pi 4t)$ .

$$y(t) = w(t) \cos(2\pi 4t) \quad \text{Use modulation property}$$

$$Y(j\omega) = \frac{1}{2} W(j(\omega - 8\pi)) + \frac{1}{2} W(j(\omega + 8\pi))$$

- (d) (10 pts) For  $x(t) = \cos(2\pi 4t)$ , the windowed signal is  $y(t) = \cos(2\pi 4t)w(t)$ . Find  $Y(j\omega)$  and sketch its magnitude,  $|Y(j\omega)|$ , below. Be sure to label all important amplitudes.

$$Y(j\omega) = \frac{1}{2} \left[ \frac{\sin((\omega - 8\pi)/2)}{(\omega - 8\pi)/2} + \frac{\sin((\omega + 8\pi)/2)}{(\omega + 8\pi)/2} \right]$$



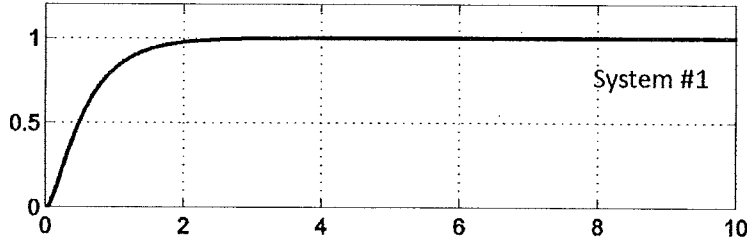
- (e) (5 pts) One goal of windowing a long signal is to use the shorter signal,  $y(t)$ , to estimate the frequency content of the longer signal,  $x(t)$ ; that is, we want  $Y(j\omega)$  to approximate  $X(j\omega)$ . Would a window length of  $T = 2$ s do a better job of estimating  $X(j\omega)$  as compared to the original window of 1s? Why or why not?

Better - the sinc pulses will be skinnier, interfere less, and be more "impulse-like".

**Problem F.5:**

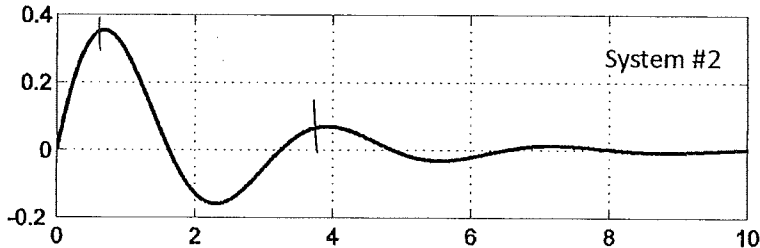
The two parts of this problem are unrelated.

- (a) (5 pts each) Consider three LTI systems, all with second order transfer functions of the form  $H(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$  (the coefficients are different for each system). A unit step is input to each system, and the three step responses shown below are measured (Systems #1, #2 and #3). For each system select a transfer function from the six provided below that is consistent with the measured step response. Put the letter (A-F) of the transfer function in the box to the right of the step response.



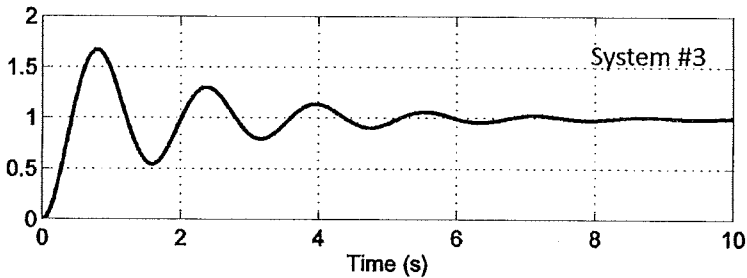
B

Transfer Function



D

Transfer Function



C

Transfer Function

$T_d \approx 3.5$   
 $\omega_d = \frac{2\pi}{T_d}$   
 $\approx 2$

A.  $H(s) = \frac{2}{s^2 + 4s + 4}$

B.  $H(s) = \frac{16}{s^2 + 10s + 16}$

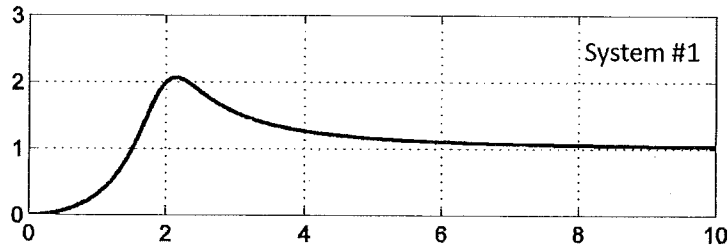
C.  $H(s) = \frac{16}{s^2 + s + 16}$

D.  $H(s) = \frac{s}{s^2 + s + 4}$

E.  $H(s) = \frac{2s}{s^2 + s + 16}$

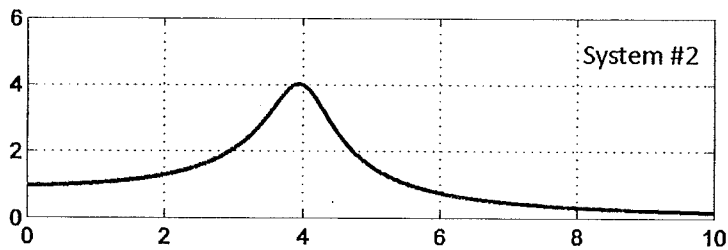
F.  $H(s) = \frac{s^2}{s^2 + 3s + 4}$

(b) (5 pts each) Let's again consider three LTI systems, which may or may not be the same as the ones in part (a), all with second order transfer functions of the form  $H(s) = \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2}$  (the coefficients are different for each system). The three frequency responses of the systems are shown below (Systems #1, #2 and #3). For each system select a transfer function from the six provided below that is consistent with the frequency response shown. Put the letter (A-F) of the transfer function in the box to the right of the frequency response.



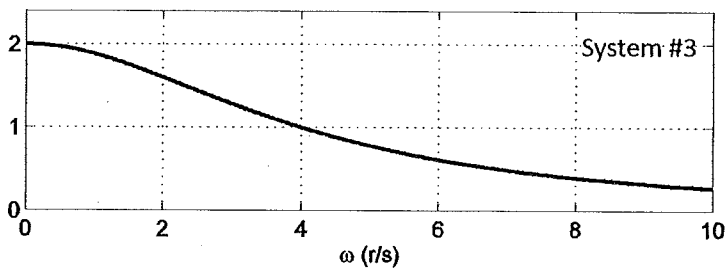
F

Transfer Function



B

Transfer Function



A

Transfer Function

A.  $H(s) = \frac{32}{s^2 + 8s + 16}$

B.  $H(s) = \frac{16}{s^2 + s + 16}$

C.  $H(s) = \frac{1}{s^2 + s + 4}$

D.  $H(s) = \frac{8s}{s^2 + 8s + 16}$

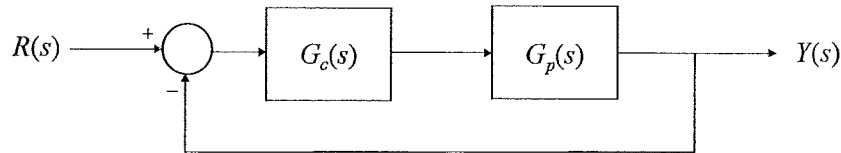
E.  $H(s) = \frac{8s^2}{s^2 + 8s + 16}$

F.  $H(s) = \frac{s^2}{s^2 + s + 4}$

**Problem F.6:**

Consider the feedback control system shown below for the following plant with two poles,

$$G_p(s) = \frac{1}{(s+1)(s+7)}$$



For parts (a), (b) and (c), use a proportional controller,  $G_c(s) = K_p$ .

- (a) (5 pts) Find the transfer function  $H(s)$  of the closed loop system in terms of  $K_p$ . Express your answer as a ratio of polynomials with all like terms combined.

$$H(s) = \frac{\frac{K_p}{s^2+8s+7}}{1 + \frac{K_p}{s^2+8s+7}} = \frac{K_p}{s^2+8s+(7+K_p)}$$

- (b) (10 pts) For  $K_p = 5$ , find the complete step response of the overall closed loop system. That is, find  $y(t)$  for  $r(t) = u(t)$ .

$$Y(s) = \frac{5}{s(s^2+8s+12)} = \frac{5}{s(s+2)(s+6)} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+6}$$

$$C_1 = \frac{5}{12}, \quad C_2 = \frac{5}{-2(-2+6)} = -\frac{5}{8}, \quad C_3 = \frac{5}{(-6)(-6+2)} = \frac{5}{24}$$

$$y(t) = \frac{5}{12}u(t) - \frac{5}{8}e^{-2t}u(t) + \frac{5}{24}e^{-6t}u(t)$$

- (c) (5 pts) What value of  $K_p$  results in a critically damped system?

$$s^2+8s+(7+K_p) = s^2+8s+16 = (s+4)^2$$

$$7+K_p = 16$$

$$\boxed{K_p = 9}$$

- (d) (10 pts) Increasing  $K_p$  from the value found in part (c) will (eventually) have which of the following effects: (circle True or False for each statement):

The steady state error for tracking a step will increase	True	False
The system will begin to oscillate	True	False
The damping ratio $\zeta$ will increase	True	False
The natural frequency $\omega_n$ will increase	True	False
The system will become unstable	True	False

For parts (e) and (f), use a PI controller,  $G_c(s) = K_p + K_i/s$ .

- (e) (10 pts) Find the transfer function  $H(s)$  of the closed loop system in terms of  $K_p$  and  $K_i$ . Express your answer as a ratio of polynomials with all like terms combined.

$$H(s) = \frac{\frac{K_p s + K_i}{s(s^2 + 8s + 7)}}{1 + \frac{K_p s + K_i}{s(s^2 + 8s + 7)}} = \frac{K_p s + K_i}{s^3 + 8s^2 + (K_p + 7)s + K_i}$$

- (f) (10 pts) Your  $H(s)$  of part (e) should have three poles. However, a PI controller only has two parameters, so you cannot place all three poles arbitrarily. Suppose you specify that one of the poles is at  $s = -2$  and that the other two poles are repeated (i.e., both are at the same unknown location). Find  $K_p$ ,  $K_i$ , and the unknown location of the repeated pole.

Assume the unknown pole is at  $s = -a$ .

The denominator should be:

$$(s+2)(s+a)^2 = (s+2)(s^2 + 2as + a^2) = s^3 + (2+2a)s^2 + (4a+a^2)s + 2a^2$$

Equate coefficients:

$$2+2a=8$$

$$a=3$$

$$K_p + 7 = 4a + a^2$$

$$= 12 + 9$$

$$K_p = 21 - 7$$

$$K_p = 14$$

$$K_i = 2a^2$$

$$K_i = 18$$