GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL & COMPUTER ENGINEERING
FINAL

DATE: 1-May-13 COURSE: ECE 3084B

NAME: Solutions STUDENT #: ______________

LAST, FIRST

- Write your name on the front page ONLY. Do not unstaple the test. You should have an additional sheet with Fourier transform tables on one side and Laplace transform tables on the other. some useful tables of Fourier transforms and properties. You may wish to detach the last page, but otherwise, do not unstaple the rest of the test.
- No calculators, laptops, phones, or other electronic devices allowed.
- Closed book. However, one page ($8\frac{1}{2}'' \times 11''$) of HAND-WRITTEN notes permitted. OK to write on both sides.
- Unless stated otherwise, justify your reasoning clearly to receive any partial credit.
- You must write your answer in the space provided on the exam paper itself. Only these answers will be graded. Circle your answers, or write them in the boxes provided. If space is needed for scratch work, use the backs of previous pages.
- The room is small for the number of students in this section. **BE CAREFUL TO NOT LET YOUR EYES WANDER.** Any sort of communication with your fellow students during this exam is strictly forbidden. Any attempt to read off of your neighbor's exam will result in unpleasant disciplinary action.
- Good luck!

Problem	*Value*	*Score*		*Problem*	*Values*	*Score*
1	20			6	20	
2	20			7	20	
3	20			8	20	
4	20			9	20	
5	20			10	20	

Problem F.1: Use Parseval's Theorem

(5 pts each) Simplify each of the following expressions.

(a) $\int_{-\infty}^{\infty}\left[\frac{\sin(10t)}{\pi t}\right]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)X^*(j\omega)\,d\omega$

call this $x(t)$

$= \frac{1}{2\pi}\int_{-10}^{10} d\omega = \frac{20}{2\pi} = \boxed{\frac{10}{\pi}}$

$X(j\omega) =$ 

$T_0 = 0.001$

(b) $\sum_{n=-\infty}^{\infty} \sin(2\pi \times 300[0.001n])\frac{\sin\left(\frac{\pi}{0.001}[t-0.001n]\right)}{\frac{\pi}{0.001}(t-0.001n)} = \boxed{\sin(2\pi\times 300t)}$

sinc interpolation formula with sample rate of 1000 Hz, supports a 300 Hz sinusoid without aliasing

(c) $\frac{d}{dt}\{\sin([3\pi/8]t)u(t+4)\} = \sin\left(\frac{3\pi}{8}t\right)\delta(t+4) + \frac{3\pi}{8}\cos\left(\frac{3\pi}{8}t\right)u(t+4)$

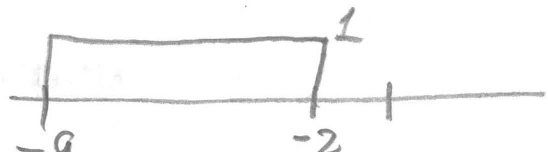
$\sin\left(-\frac{3\pi}{2}\right) = 1$

$= \boxed{\delta(t+4) + \frac{3\pi}{8}\cos\left(\frac{3\pi}{8}t\right)u(t+4)}$

$\frac{3\pi}{8}\times 4 = \frac{-3\pi}{2}$

(d) $\int_{t-3}^{t+4}\delta(t+5)dt = u(t+5)\Big|_{t-3}^{t+4} = \boxed{u(t+9)-u(t+2)}$

or you could draw a plot:



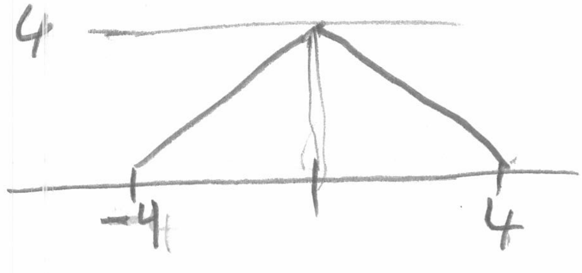
Remember autocorrelation functions are symmetric!

Problem F.2:

The three parts of this problem are unrelated to each other.

$x(t)$ has unit height and is 4-time units long.

(a) (5 pts) **Draw a labeled sketch of the *autocorrelation function*** of $x(t) = u(t-3) - u(t-7)$, i.e. the correlation of $x(t)$ with itself.

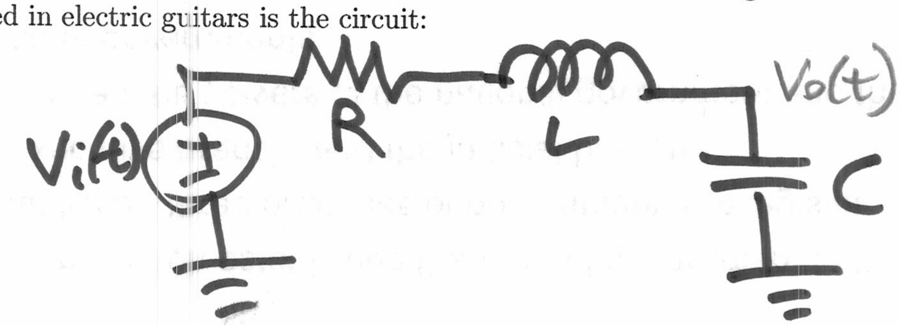


(b) (5 pts) **Find** $X(s)$, the **Laplace transform** of $x(t) = e^{-3t}u(t-4)$, using your handy dandy table of Laplace transforms and properties. (This may require a slight degree of cleverness).

$x(t) = e^{-3(t-4)} e^{-12} u(t-4)$ ← manipulate expression so you can use the delay property

$$\boxed{X(s) = e^{-12}\frac{e^{-4s}}{s+3}}$$

(c) (10 pts) A reasonable approximation to the behavior of a single-coil electromagnetic pickup used in electric guitars is the circuit:



Use a voltage divider

Find $H(s)$, **the transfer function** that relates the "input" $v_i(t)$ to the "output" $v_o(t)$.

$$V_o(s) = V_i(s)\underbrace{\frac{\frac{1}{sC}}{Ls + R + \frac{1}{sC}}}_{H(s)}$$

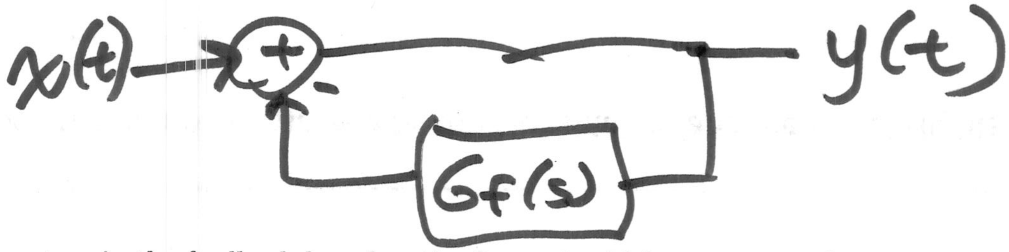
$$H(s) = \boxed{\frac{1}{LCs^2 + RCs + 1} \text{ or } = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}$$

If needed, you can always rederive the needed feedback formula: $Y(s) = X(s) - Y(s)G_f(s)$

$$\frac{Y(s)}{X(s)} = \frac{1}{1+G_f(s)}$$

Problem F.3:

The Buchla Model 191 Sharp Cutoff Filter contains some rather unusual circuitry. One of its circuits implements this strange negative feedback system:



The system in the feedback loop has the first-order highpass system function

$$G_f(s) = \frac{10s}{s+a}, \quad \text{where } a > 0,$$

and, yes, there is indeed *nothing* in the "feed-forward" part.

(a) **Find the closed-loop system** function $H(s)$ that relates the output $y(t)$ to the input $x(t)$.

$$H(s) = \frac{1}{1+G_f(s)} = \frac{1}{1+\frac{10s}{s+a}} = \frac{s+a}{s+a+10s}$$

$$= \boxed{\frac{s+a}{11s+a} \text{ or } \frac{1}{11}\left(\frac{s+a}{s+\frac{a}{11}}\right)}$$

(b) **Find the D.C. gain** of the closed-loop system, i.e. the frequency response for a frequency of 0.

$$H(j0) = \frac{a}{a} = 1$$

(c) **What does frequency response** $H(j\omega)$ converge to as $\omega \to \infty$?

$$\lim_{\omega\to\infty} H(j\omega) = \lim_{\omega\to\infty} \frac{j\omega}{11j\omega} = \frac{1}{11}$$ (the "a" constants get swamped)

(d) $H(j\omega)$ isn't *strictly* a highpass or lowpass filter as we defined the terms in class. But based on your answers to (c) and (d), **which term, "highpass" or "lowpass" do you think more accurately describes the behavior of** $H(j\omega)$?

lowpass

Problem F.4:

An inverted pendulum can be roughly modeled using the plant system function

$$G_p(s) = \frac{-s^2/L}{s^2 - g/L} = \frac{-s^2}{s^2-1} = \frac{-s^2}{(s-1)(s+1)}$$

where the input of the system, $x(t)$, is the position of a cart the pendulum is sitting on, and the output of the system, $y(t)$ is the angle of the pendulum in radians, L is the length of the pendulum, and g is the acceleration due to gravity. Deriving this system function is really messy, so you can take it on faith. It relies on making small-angle approximation to trigonometric functions; we will assume that is valid.

To make our lives easier, let's assume $L = 1$ meter, and let's also assume that we are working on a planet where the gravity is $g = 1$ meter/sec^2, unlike the 9.8 meter/sec^2 we have on Earth. (See how nice that makes $G_p(s)$?)

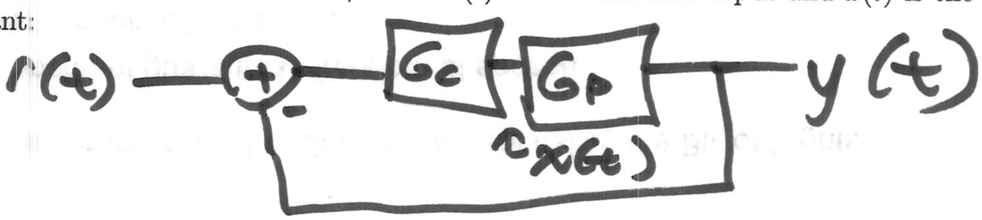
(a) (5 pts) **Find the differential equation** that relates the input $x(t)$ to the output $y(t)$. Please place all terms with $y(t)$ on the left hand side and all terms with $x(t)$ on the right hand side.

$$\boxed{\ddot{y}(t) - y(t) = -\ddot{x}(t)}$$

(b) (5 pts) **Find the poles** of $G_p(s)$. Is $G_p(s)$ **BIBO stable?** If not, **tell me,** which pole is the troublemaker?

$s_p = \pm 1$; not BIBO stable because of pole at $+1$

(c) (10 pts) Let's try a PI controller with the system function $G_c(s) = (K_p s + K_i)/s$ in the following negative feedback control scheme, where $r(t)$ is the reference input and $x(t)$ is the input to the plant:



Find the K_p and K_i that would result in the closed-loop system having a **critically damped** closed-loop response with a **natural frequency** of $\omega_n = 2$ radians per second.

$$H(s) = \frac{\left(\frac{K_p s + K_I}{\cancel{s}}\right)\left(\frac{-s^{\cancel{2}}}{s^2-1}\right)}{1+\left(\frac{K_p s + K_I}{\cancel{s}}\right)\left(\frac{-s^{\cancel{2}}}{s^2-1}\right)} = \frac{-K_p s^2 - K_I s}{s^2 - 1 + K_p s^2 - K_I s}$$

$$= \frac{-K_p s^2 - K_I s}{s^2(1-K_p) - K_I s - 1} = \frac{\frac{-1}{1-K_p}\left(K_p s^2 + K_I s\right)}{s^2 - \frac{K_I}{1-K_p} s - \frac{1}{1-K_p}}$$

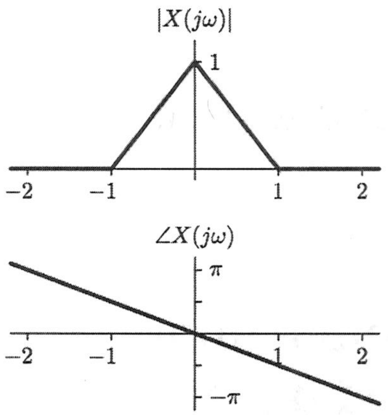
$$\frac{1}{K_p - 1} = \omega_n^2 = 4 \Rightarrow K_p - 1 = \frac{1}{4} \Rightarrow \boxed{K_p = \frac{5}{4}}, \quad \frac{K_I}{K_p - 1} = 2\zeta\omega_n = 4, \quad \zeta = 1$$

$$\frac{K_I}{1/4} = 4 \Rightarrow \boxed{K_I = 1}$$

Problem F.5:

(4 pts each) *This problem is adapted from Problem 4 on Quiz 3 from the Fall 2009 offering of MIT's 6.003: Signals and Systems. The images shown are screenshots of the PDF of that quiz.*

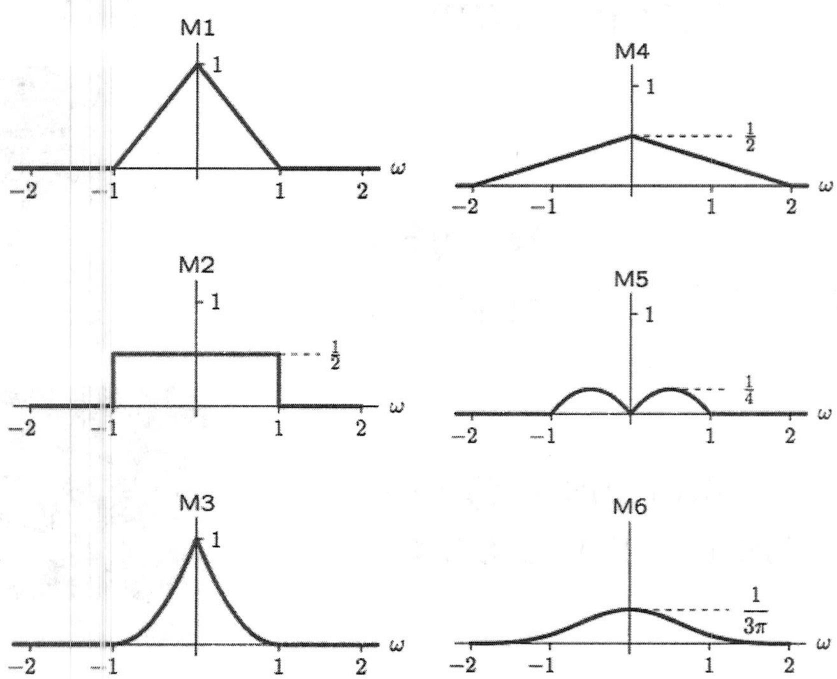
The magnitude and angle of the Fourier transforms of a signal $x(t)$ are specified by:



Five signals $y(t)$ are derived from $x(t)$, as shown immediately below. Six Fourier-domain magnitude plots (M1-M6) are shown at the bottom of the page. Determine which of these plots is associated with each of the derived signals, and **write the correct label** (e.g., M1) on the line next to each signal. Note that each plot will be used at most one, so one plot is not used. The asterisks in part (b) denotes convolution.

a) $y(t) = \frac{dx(t)}{dt}$: M5 b) $y(t) = x(t) * x(t)$: M3 c) $y(t) = x(t - \pi/2)$: M1

d) $y(t) = x(2t)$: M4 e) $y(t) = x^2(t)$: M6

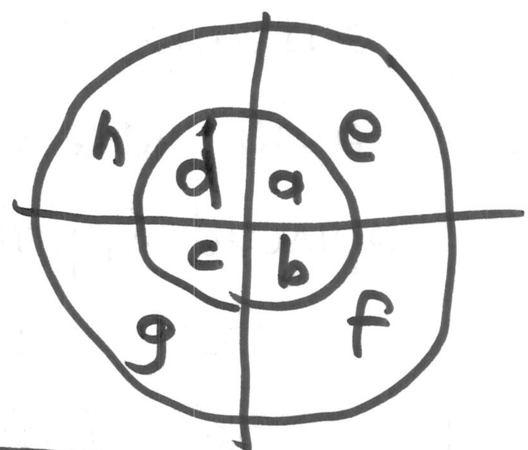


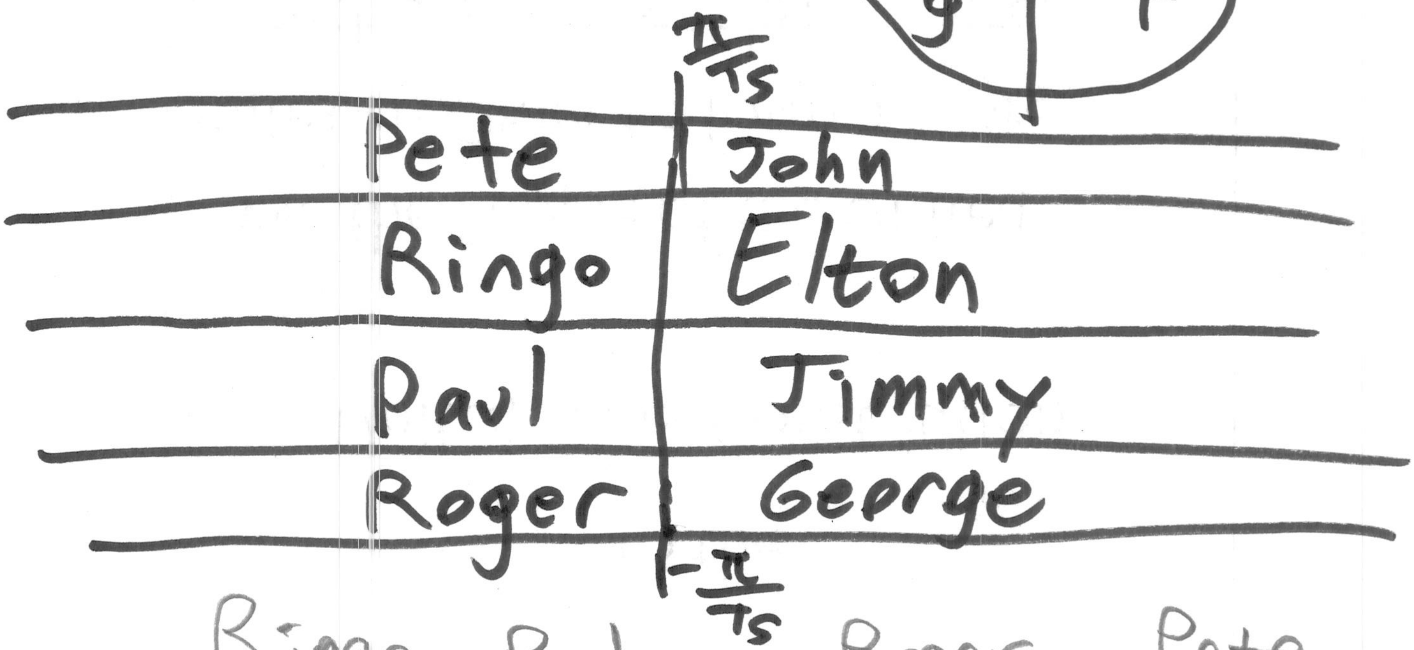
Problem F.6:

In class, we explored the Laplace transform of continuous-time signals sampled with a sampling period of T_s. We showed that points in the s-plane in a horizontal strip whose imaginary values lie between $-\pi/T_s$ and π/T_s can be mapped to the z-plane using the relation $z = e^{sT_s}$.

The s-plane and z-plane are shown below. Each plane is divided into eight different regions. The regions in the s-plane are given letters, an the regions in the z-plane are given the names of British rock stars from the 1960s and 1970s. **Match the regions** in the s-plane with the corresponding region in the z-plane under the mapping $z = e^{sT_s}$ by **writing the appropriate rock star's name** next to the letters at the bottom of the page.

(Don't get too wrapped up in thinking about all the complicated filter design work we did in class and on the homework. If you truly understand how e-to-the-whatever works, you should be able to do this problem even if you missed that lecture.)





a) Ringo b) Paul c) Roger d) Pete

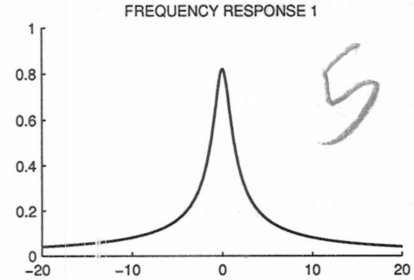
e) Elton f) Jimmy g) George h) John

Prob. # 7 Problem by Prof. Ivan Selesnick

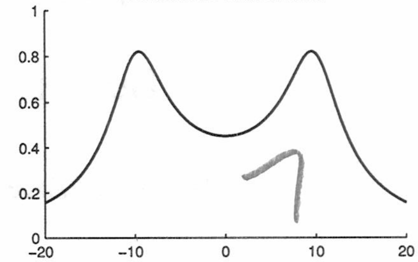
The frequency responses of eight causal continuous-time systems are illustrated below, along with the pole/zero diagram of each system. But they are out of order. Match the figures with each other by completing a table.

FREQUENCY RESPONSE	POLE-ZERO DIAGRAM
1	5
2	7
3	2
4	1
5	3
6	4
7	8
8	6

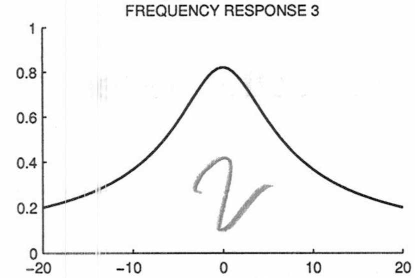
FREQUENCY RESPONSE 1



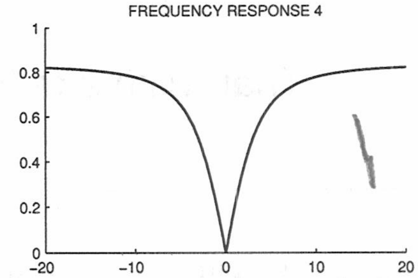
FREQUENCY RESPONSE 2



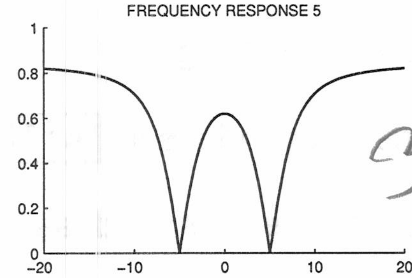
FREQUENCY RESPONSE 3



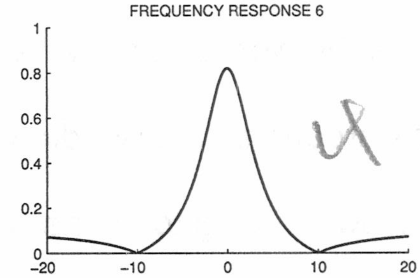
FREQUENCY RESPONSE 4



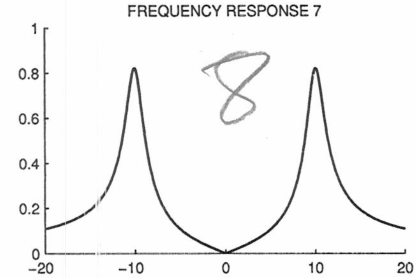
FREQUENCY RESPONSE 5



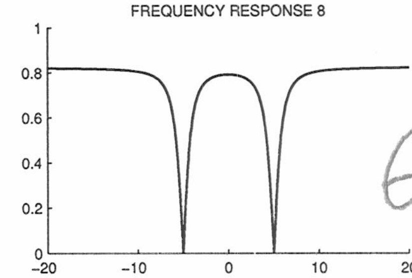
FREQUENCY RESPONSE 6



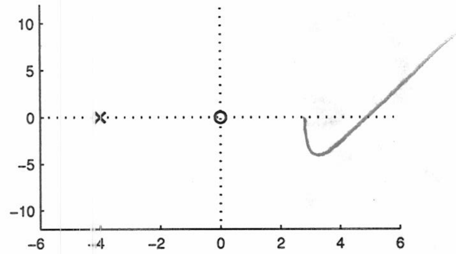
FREQUENCY RESPONSE 7



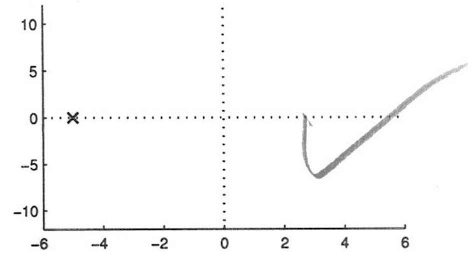
FREQUENCY RESPONSE 8



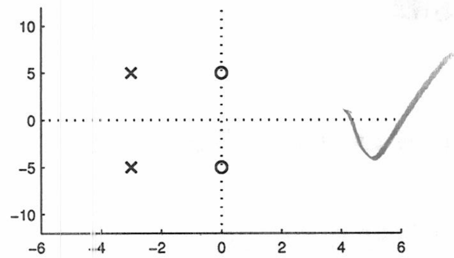
POLE–ZERO DIAGRAM 1



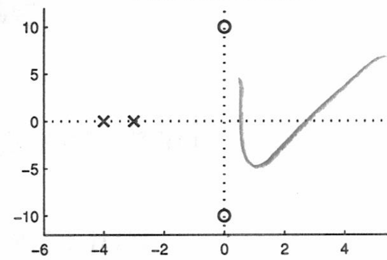
POLE–ZERO DIAGRAM 2



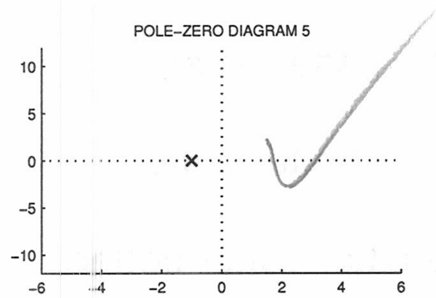
POLE–ZERO DIAGRAM 3



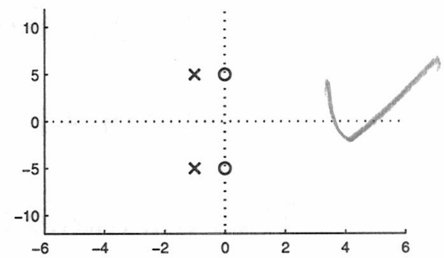
POLE–ZERO DIAGRAM 4



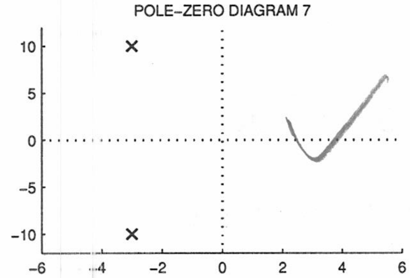
POLE–ZERO DIAGRAM 5



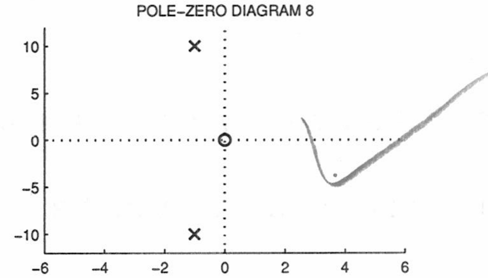
POLE–ZERO DIAGRAM 6



POLE–ZERO DIAGRAM 7



POLE–ZERO DIAGRAM 8

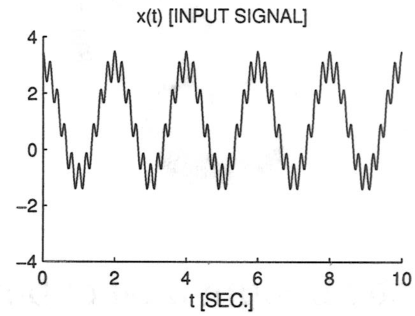


Prob. 8 Problem by Prof. Ivan Selesnick

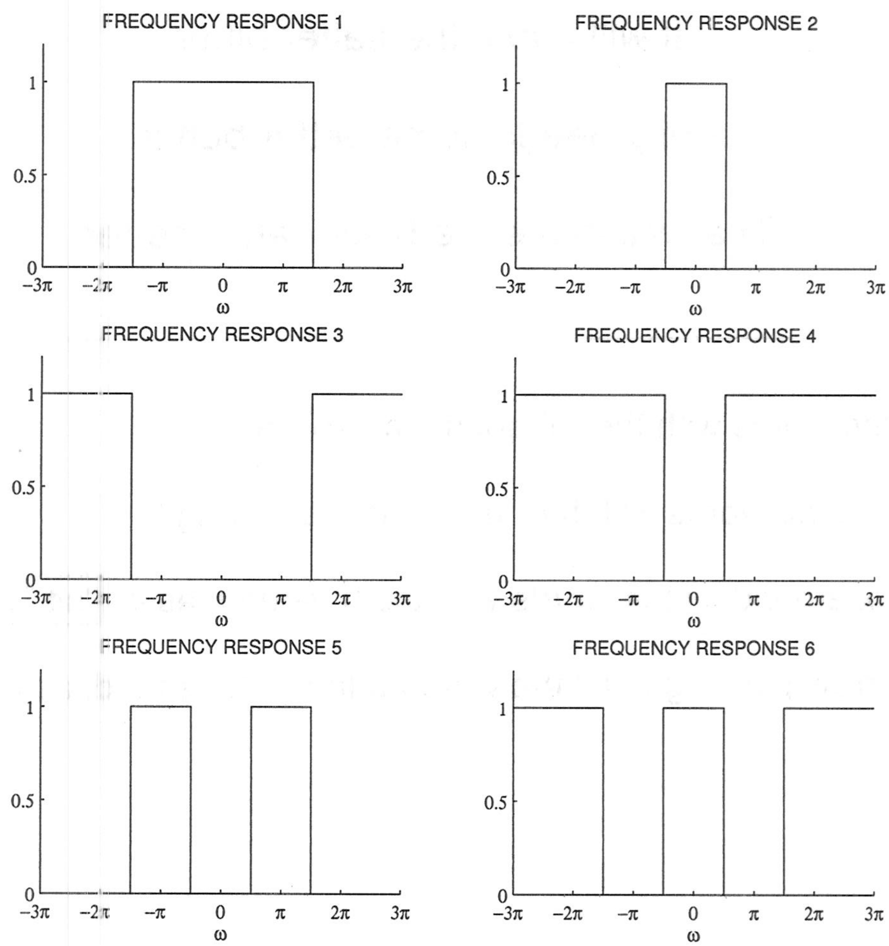
A signal $x(t)$, comprised of three components,

$$x(t) = 1 + 2\cos(\pi t) + 0.5\cos(10\pi t)$$

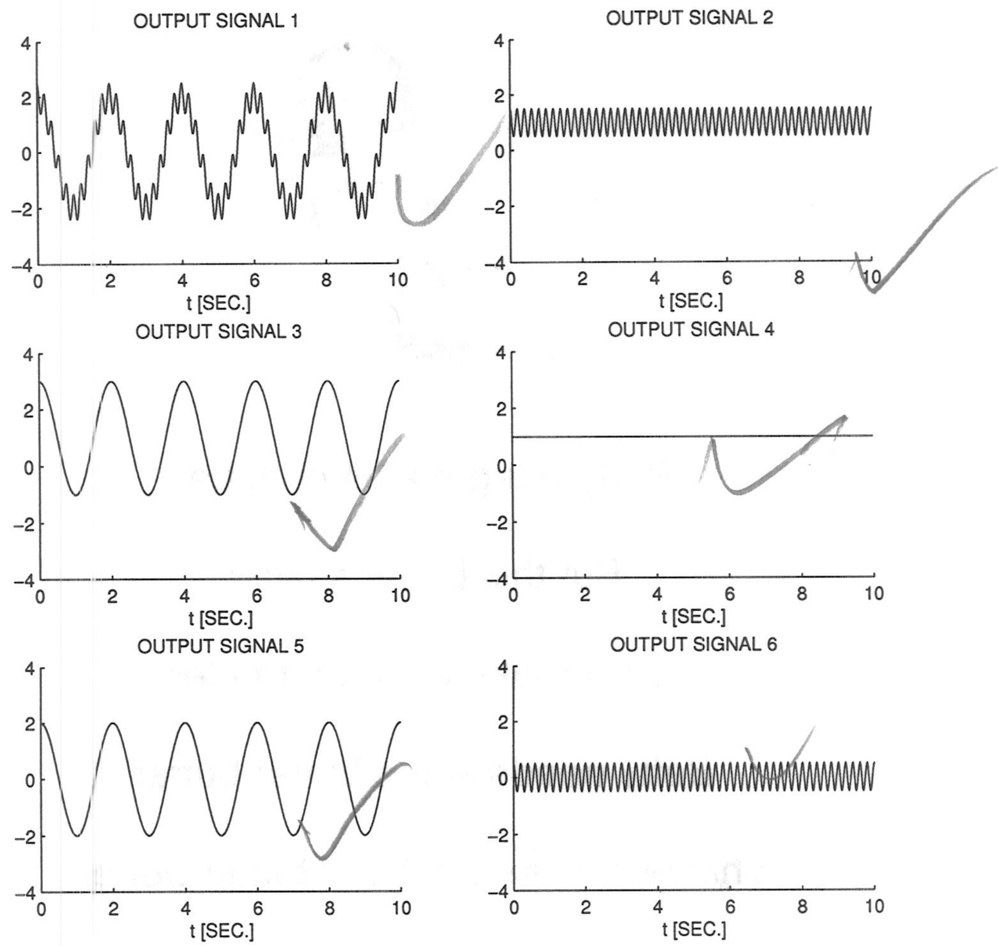
is illustrated here:



This signal, $x(t)$, is filtered with each of six different continuous-time LTI filters. The frequency response of each of the six systems are shown below. (For $|\omega| > 3\pi$, each frequency response has the value it has at $|\omega| = 3\pi$.)



The six output signals are shown below, but they are not numbered in the same order.



Match each output signal to the system that was used to produce it by completing the table.

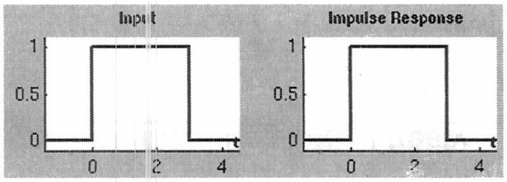
System	Output signal
1	3
2	4
3	6
4	1
5	5
6	2

Problem F.9:

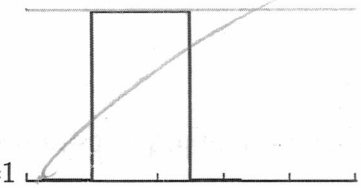
The lettered images on the next page show impulse responses next to inputs provided to the system with the given impulse response. The numbered images images show outputs. Match the input/impulse response pairs on the left with the outputs on the right by **writing the number of the corresponding output by the graph of the corresponding input/impulse response pair**. Note that while **the horzizontal axes of the outputs all have the same spacing, the spacing of the input and impulse responses changes, so be careful to read the numbering on the axes of the input graphs.** The vertical axes of the outputs may change from plot to plot.

You do not need to provide any explanations. **This is not an equation crunching sort of problem; use your intuitive "feel" for convolution.**

9



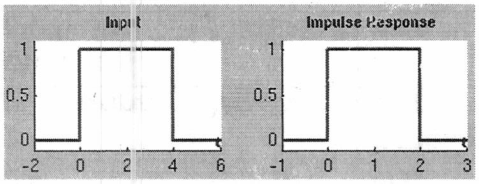
(a)



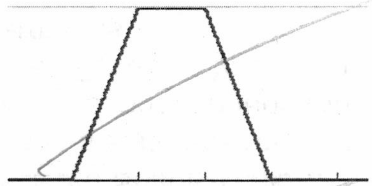
#1

g

2



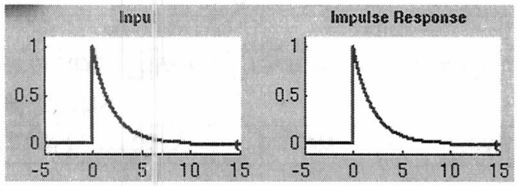
(b)



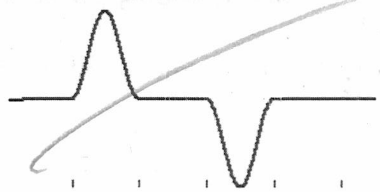
#2

b

7



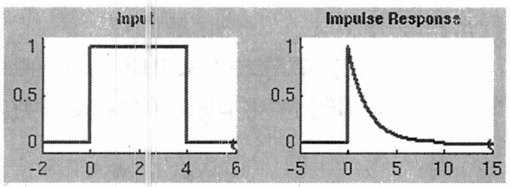
(c)



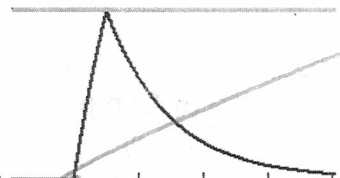
#3

j

10



(d)



#4

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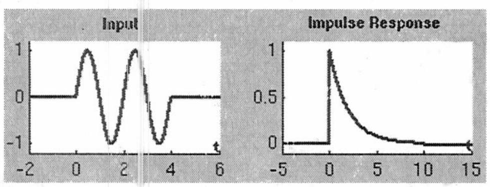
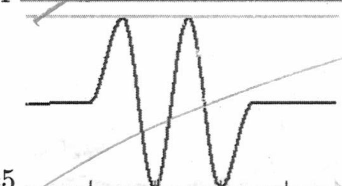
6



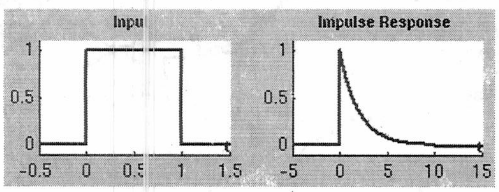
(e)



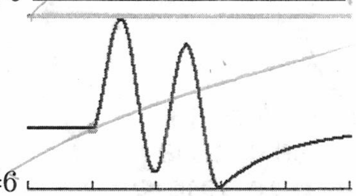
#5

h

4



(f)



#6

e

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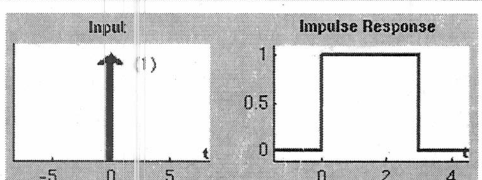

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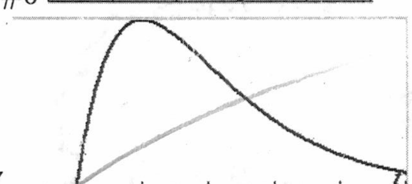

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c

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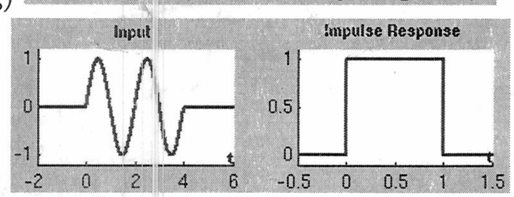

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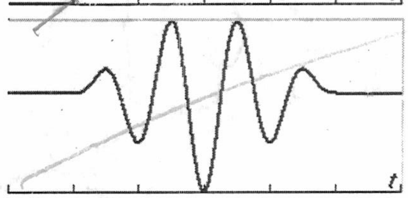

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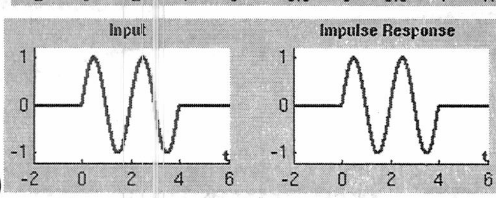

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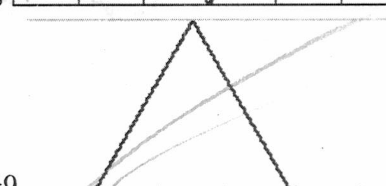

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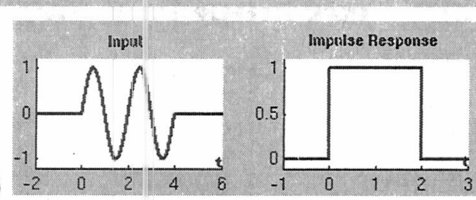

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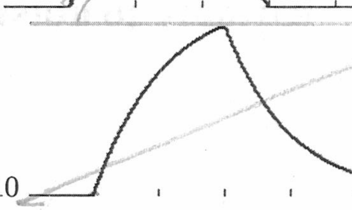

#10

d

Problem F.10:

The following two pages were taken from a handout labeled "Tutorial 10" from the Spring 2004 offering of the MIT's 6.003: Systems and Signals. The first of those two pages shows eight pole-zero s-plane plots. The second of those two pages shows eight **step** responses.

Match the pole-zero plots with the step responses by **writing the letter of the step response on the associated pole-zero plot.**

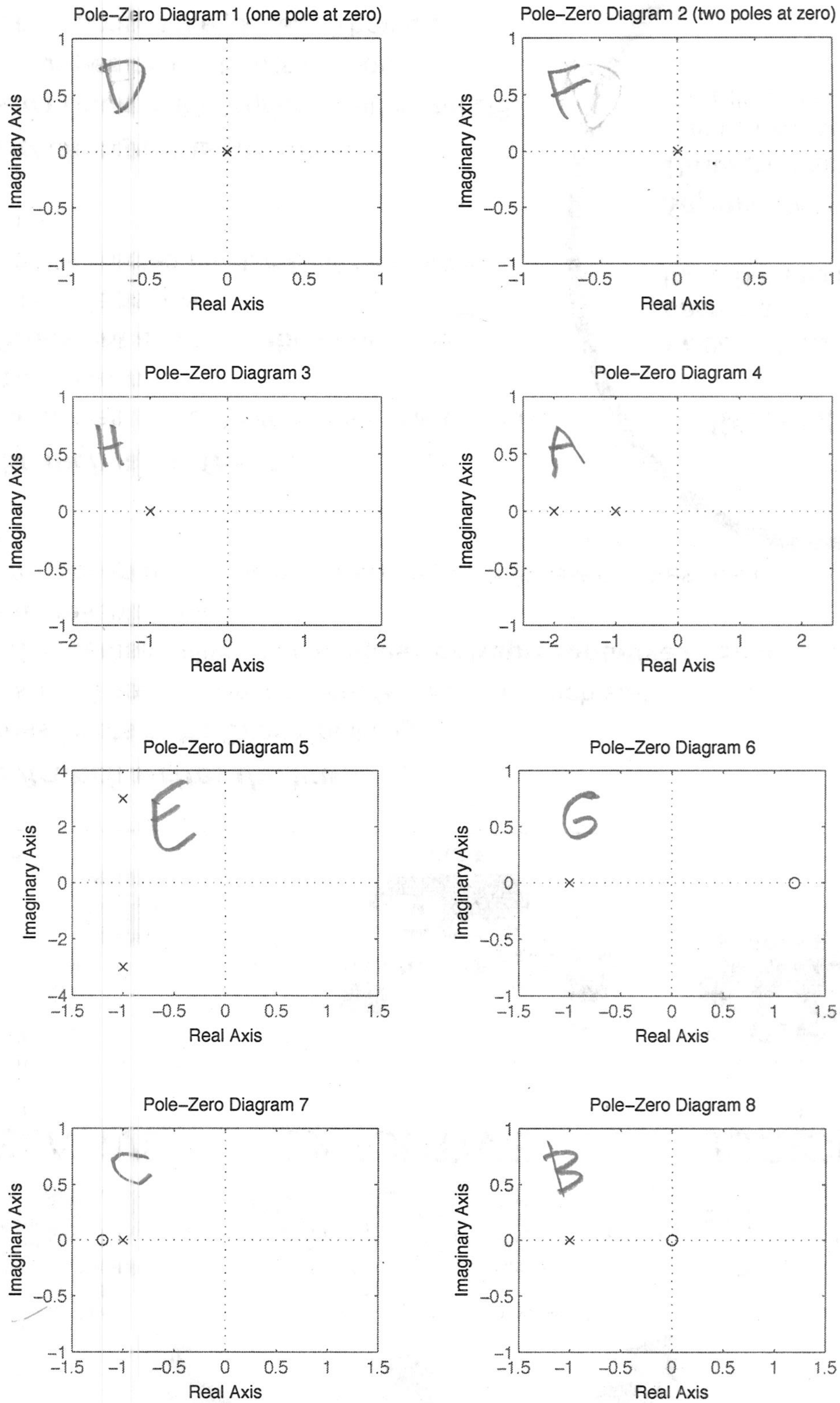

Figure 1: Matching Problem: Pole-Zero Diagrams

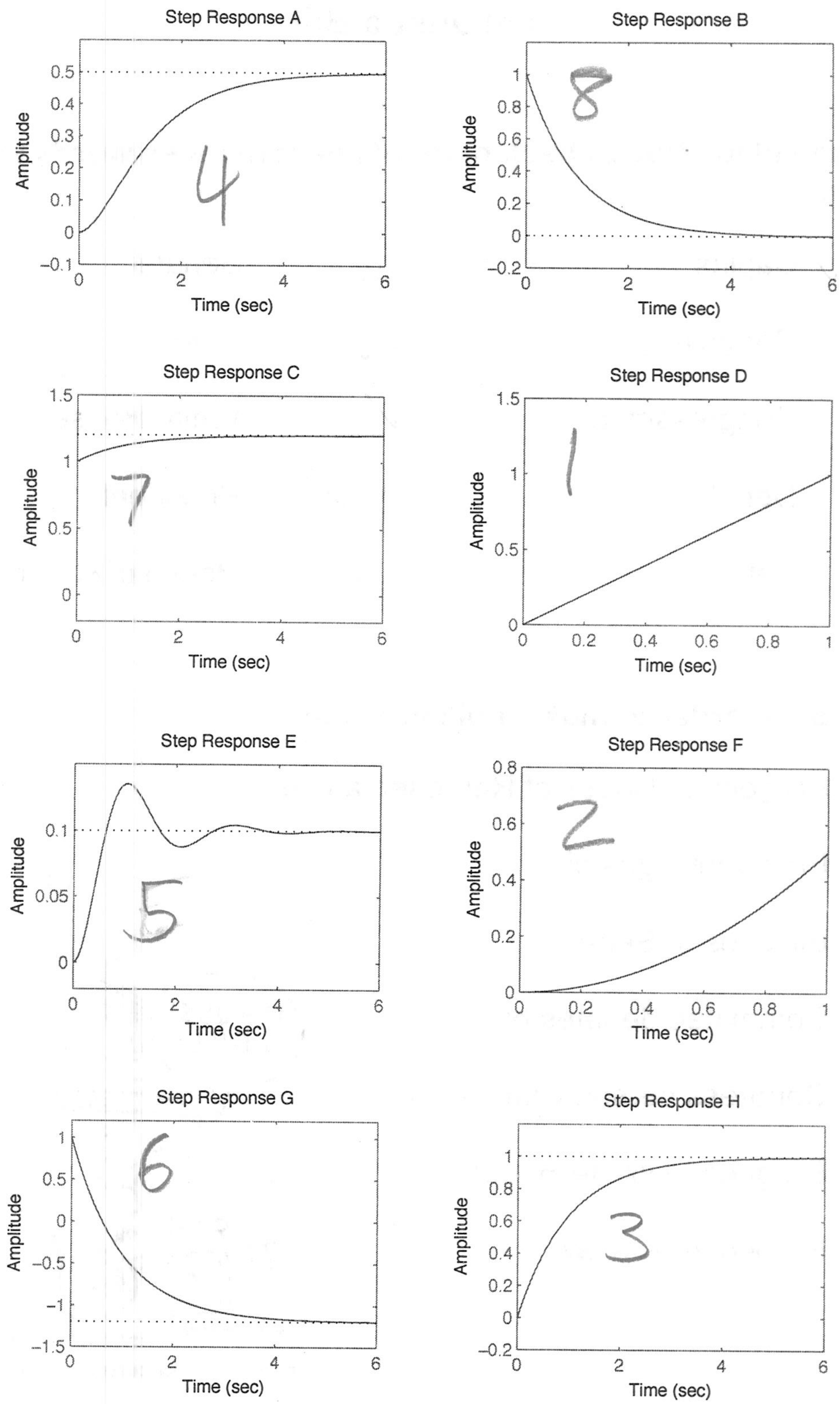

Figure 2: Matching Problem: Step Responses